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Evaluation and extension of the potential for application of the behavioural framework to practical engineering problems

Pfaff, Raphael

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Evaluation and Extension of the Potential for Application of the Behavioural Framework to Practical Engineering Problems

Raphael Pfaff
BSc Mathematics, Dipl.-Ing. (FH) Mechatronics (Germany)
MSc (Distinction) Control Engineering (United Kingdom)

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Control Theory and Applications Centre, Coventry University

Abstract

Motivated by the theoretical advantages of the Behavioural Framework for modelling and control, which is based on model formulations without preconception of the input/output structure of the system, this thesis sets out to investigate the applicability of this framework to practical control engineering problems. While the Behavioural Framework is consistent and all-encompassing, certain areas are identified where more practically oriented approaches are likely to improve the applicability and thus increase the popularity of the Behavioural Framework. In these areas, novel techniques are developed and tested, extending the Behavioural Framework with immediately applicable techniques.

The development, documentation and exchange of models based on physical principles forms the basis for many practical problems in modelling and control. This topic is addressed with a view on model validity and graphical representations, both for documentation and exchange as well as as input for simulation software.

The validity of a model can be increased further by taking into account system nonlinearities. Commonly used nonlinear model classes are analysed with application in the Behavioural Framework in mind and the best suited class, the class of bilinear systems, is represented, analysed and tested on practical problems.

The identification of systems from recorded data, termed approximate modelling, is important in modelling and control. Here a novel practically viable approach, the combined misfit/complexity approach, is presented. This approach weighs the model misfit versus the model complexity and in this way resembles a heuristic approach.

A novel technique for adaptive control in the Behavioural Framework is developed based on the properties of practical control systems, among these

are the mixed time axis and the limited availability of controller representations. The scheme, based on a recursive Errors-in-Variables estimator and an interconnected controller, is applied to linear time varying as well as bilinear plants.

The application of existing and novel techniques to practical control engineering problems forms a *leitmotif* of this thesis.

Acknowledgement

The research presented in this thesis was life-enhancing through constant challenge, it would not have been possible without help and inspiration by many people.

I am indebted to my supervisor, Prof. Keith Burnham, who challenged me in the spring of 2006, during my MSc time at CTAC, with the Behavioural Framework. This challenge over the course of my PhD changed for inspiration and support in always too short discussions in Coventry. Dr. Tomasz Larkowski frequently took part in these discussions and dedicated part of his time to share valuable advice and views on nonlinear systems and system identification.

Thanks are due to the CTAC team, for bearing with me as an occasional guest and providing a creative working atmosphere. I also thank Piotr and Paulina Sydor, for offering me a home in Coventry and sharing views on life in general and PhD life in particular.

I wish to thank my wife Tina and children Henri and Josefine for making our house worth living in and in this way fostering creativity as well as bearing with my absence.

Lastly, I thank Prof. Jan C. Willems, for his revolutionary views on systems and control, that are well worth the years I spent on this PhD thesis.

To Tina, Henri and Josefine

[Mais] dieu a choisi [le monde] qui est le plus parfait, c'est-á-dire celui qui est en même temps le plus simple en hypothses et le plus riche en phenomenes.

God has chosen [the world] that is the most perfect, that is to say, the one that is at the same time the simplest in hypotheses and the richest in phenomena.

Gottfried Wilhelm von Leibniz (1646 - 1716)
Discours de metaphysique (1686)

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Chapter 1

Introduction and Motivation

> Mathematicians are [like] a sort of Frenchmen; if you talk to them, they translate it into their own language, and then it is immediately something quite different.
>
> Johann Wolfgang von Goethe

1.1 Introduction

The last decades have seen the advent of a novel framework for modelling and control, termed the *Behavioural Framework* as coined by Jan C. Willems (Willems, 1986a). The Behavioural Framework is remarkable since it puts into question the *a priori* distinction of system variables into inputs and outputs and shares the same foundation on set theory as modern mathematics. This framework has been developed further since its first presentation, however has seen little attention in practical application.

Modelling in the Behavioural Framework means to find a law that decides whether a given time trajectory is a possible outcome of the system. In this way, behaviours (i.e. sets of possible time trajectories) are mapped to the system, which marks a way of model definition without *a priori* distinction between input and output. The models gained in this way are acausal by nature, a paradigm of modelling which has found increasing interest in the community (Abel, 2010). An indication for the growing interest in acausal modelling is the increasing number of software tools for acausal modelling and simulation, mostly based on the Modelica language. Additionally, techniques for recuperation of energy foster the interest in modelling of systems that invert their energy flow direction during operation, an application where modelling in terms of transfer functions has drawbacks.

While the application of energy recuperation is supposed to create a large initial leap in efficiency even at comparably low efficiency levels, it can be expected that later optimisation cycles will require more appropriate control. The usage of feedback control systems provides appropriate control in terms of input/output relation, however an adequate acausal control law may yield further potential for improvement. A framework for acausal control is provided by the Behavioural Framework through the concept of control by interconnection. Control by interconnection is achieved by means of an appropriate control system, interconnected to the plant, attaining an overall behaviour as desired. While applications of acausal modelling are increasing, usage of acausal control techniques appears to be lagging behind in terms of numbers.

This thesis sets out to investigate the potential of the Behavioural Framework for application to practical systems and, where techniques are missing, extends the existing techniques.

1.2 Motivation

The motivating factors for this evaluation are not only rooted in the Behavioural Framework itself, rather it can be found that the Behavioural Framework reinforces already present advantages that modelling and control can offer. It has become a standard to treat signals and systems in terms of model representations, which are frequently biased by our interest or the intended purpose of the model (Albertos and Mareels, 2010).

Mathematical modelling is widely accepted to reduce product development cycles and to increase safety of product development (Isermann, 2008). Further to their application in early stages as simulation in lieu of the prototypes, mathematical models also help to gain understanding of a system in question and can effectively lead to mastering, refinement and extension of technologies. For this reason, engineers and mathematicians alike have been applying models, not only of technical systems, for centuries.

The gain in the understanding of a system can be increased when approaching the modelling process without an *a priori* conception of the input/output structure of the system, an approach which is promoted in the Behavioural Framework. This preconception of the input/output structure, constituting the causality of the system, is mostly the result of previous analytical modelling or assumptions.

The application of control to a system is typically motivated by the need to adapt the behaviour of the to-be-controlled-system in order to achieve an increased accuracy, compatibility, safety or repeatability. The process of control design is frequently based on a model of the system, which makes it possible to simulate the control performance of the controlled system.

In an input/output framework, control is mostly achieved by the use of feedback control systems. In its application to control, the Behavioural Framework increases the control engineers set of techniques by formulating control without feedback through introduction of the concept of control by interconnection. This type of control can, in many applications, prove to be a simple and robust control. Control by interconnection is frequently achieved by utilising energy in the system, in contrast to powering additional actuators.

In addition to the motivations the Behavioural Framework brings through its promotion of advantages of the field of modelling and control, the fact that it forms a disruptive innovation of the field while at the same time being founded on appealing mathematical concepts, makes the Behavioural Framework itself an attractive subject for study. Indeed, Jan C. Willems is considered to be a 'thought provoking and inspiring teacher' (Albertos and Mareels, 2010, p. 17). An especially important motivation of this thesis is thus the identification of hindrances in teaching and application of the Behavioural Framework.

A final equally important source of motivation stems from the need for extension of the Behavioural Framework in certain areas. The areas in need for extension are mainly those where the control engineer wishes to find tools and techniques of immediate applicability to practical problems, such as simulation, empirical modelling, nonlinear modelling and control techniques. While the Behavioural Framework is an all encompassing and consistent framework, there are certain aspects that require a succinct form, together with some tools, to find an increasing number of users and applications to practical problems.

1.3 Problem statement

A practical control problem is distinct from its theoretical counterparts mainly by its degree of complexity, its nonlinearity or time invariance, its continuous time axis and the users inability to observe the noise free signals of the system. Many of these differences may be omitted for applications to certain systems, however sometimes the omission leads to suboptimal modelling and control performance. At the same time, the assumption that a system is linear, time-invariant and signals may be recorded without noise effects is common to many works in the Behavioural Framework. Similarly, many concepts are developed under the assumption of a common time axis for all system components, which does not hold true for computer control systems applied to real world plants.

The above said, the problem addressed in this thesis is to analyse the potential of the Behavioural Framework for application to such practical (control) engineering problems and where extensions may increase the applicability to derive such approaches.

In the context of this thesis, this problem can be reduced to a set of subproblems, reflecting typical control engineering tasks:

- Analytical modelling: derive models from physical principles and similar principles in the respective discipline.
- Integration of graphical model representations: represent models and submodels in graphical form for documentation and exchange.
- Model validity: Find limits of validity of certain model structures that prevent useful models from being derived, where necessary extend the model validity.
- Approximate modelling: analyse existing algorithms and find a more practical approach to approximate modelling.
- Adaptive control: analyse existing approaches to adaptive control in the Behavioural Framework and devise means to gain a more practical approach.
- Control engineering education: identify additional material to be included in the control engineering curricula in order to ease access to the Behavioural Framework.

The analysis and extension of the potential for application of the Behavioural Framework goes hand in hand with the application of existing and novel techniques to selected practical control problems.

1.4 Outline of approach

1.4.1 Structure of thesis

This thesis follows a linear development of topics, taking into account that applications of control to systems are rarely performed without prior modelling. In this order, initially selected mathematical and scientific preliminaries are laid. On this basis, a literature survey of the Behavioural Framework and other fields relevant for this thesis is presented. The further development of topics starts with analytical modelling, is continued via the representation of certain classes of nonlinear systems and approximate modelling (i.e. system identification) to finish with adaptive control.

1.4.2 Chapter outlines

Chapter 2 provides mathematical and other scientific background as well as a brief historical view and an introduction to the Behavioural Framework. Especially the control engineering part is presented in order to re-address some topics which are commonly known, but typically include some bias towards the input/output framework. The mathematical contents of this chapter represent some mathematical knowledge, which is common for mathematicians, but rarely taught in this way to engineering students. The historical development of the field helps to understand when and why the *a priori* distinction between input and output ports was introduced to the field of control.

Chapter 3 contains a literature survey, including literature from the Behavioural Framework, nonlinear systems and simulation software, as well as some interdisciplinary topics. The section featuring literature from the Behavioural Framework presents the foundational publications and more recent developments such as control and adaptive control in the Behavioural Framework. While this thesis aims to explore the practical side of the Behavioural Framework, some articles highlighting the reception in the field of applied mathematics are also given. The review of literature from the field of nonlinear systems introduces work on some well applicable classes of nonlinear models, which in the sequel are analysed for their applicability in the Behavioural Framework. A review of simulation software for applicability in the Behavioural Framework and a brief view on related interdisciplinary topics finish the chapter.

Chapter 4 deals with analytical modelling in the Behavioural Framework, based on a view of model validity. In this context, graphical model representations are compared and some acausal linear first principles modelling examples are provided. The application study on the thermal behaviour of a brake disc for rail vehicles concludes this chapter.

Chapter 5 presents the extension of the model validity by taking into account nonlinearities. For this purpose, the properties of selected subclasses of nonlinear systems is prepared and a suitable class, the class of bilinear systems, further developed and brought to applications on hydraulic and chemical systems.

Chapter 6 introduces system identification, termed modelling, in the Behavioural Framework. The development is followed from exact to approximate modelling, introducing the terminology and concepts. Based on this, a practical approach is developed and presented as an algorithm. This algorithm is tested utilising a Monte Carlo simulation and applied to a nonlinear continuous time model of a CSTR.

Chapter 7 develops a scheme for adaptive control in the Behavioural Framework, based on a set of assumptions aiming to reflect practical limitations of control systems. The problem of adaptive control in the Behavioural Framework is reduced to a number of subproblems, for which subsequently solutions are proposed. The proposed scheme is tested on a discrete time system in a Monte Carlo study and applied to several second order continuous time systems of linear time varying and nonlinear nature.

1.4.3 Methodology

The research question addressed in this thesis, the question for the applicability of the Behavioural Framework to practical control engineering problems, has to be answered by practical problems. In this sense, the application of existing and novel behavioural techniques to such problems plays a major role.

The practical problems are represented by computer models simulated in appropriate simulation software, such as Matlab and MapleSim. Taking into account the practical nature of the simulations, the results are mostly analysed qualitatively. The computer models are documented such that an implementation based on the information is feasible.

Novel techniques are developed by extension or combination of existing methods, in this way the existing approaches serve as a foundation for the novel techniques. In the development of novel techniques, tests are performed, mostly based on Monte Carlo simulations. The novel techniques are described in algorithmic or textual form, so that implementation and understanding is simplified.

A relatively minor part is played by the concept of mathematical proof. In this case, the truth of a given statement is formally proved based on other theorems. Occasionally, in the development of arguments, other forms of deductive or inductive reasoning is applied.

1.5 Contributions

The research related to this thesis has led to contributions to the body of knowledge, which can be grouped into major and minor contributions. The major contributions and the related publications are:

- Representation and analysis of bilinear systems in the Behavioural Framework: a suitable representation for bilinear systems in the Behavioural Framework, the so called bilinear extended kernel representation, is derived, the existence and uniqueness of the solutions to the respective behavioural equations is shown and the practical applicability of the novel representation is tested. Related publication:
  - Pfaff, R. and Burnham, K. (2008b). Representations of bilinear systems in the behavioural framework with application to a continuous stirred tank reactor. In *Proc. of the 6th European Workshop on Advanced Control and Diagnosis, Coventry, UK*
- Development of a combined misfit/latency approach to approximate modelling: a combined approach, similar to that proposed by (Willems, 1987), is developed and tested on practical modelling problems. Related publication:
  - Pfaff, R. and Burnham, K. (2009a). Bilinear system identification in the behavioural framework. In *Proc. of the 20th Int. Conference on Systems Engineering, Coventry, UK*
- Development of a practically viable scheme for adaptive control in the Behavioural Framework: a scheme for adaptive control is developed, tested in comparison to other approaches, tested on practical mixed time axis systems and implemented ready for use on practical systems. Related publications:

  - Pfaff, R., Larkowski, T., and Burnham, K. (2010). An approach to adaptive control in the behavioural framework. In *Proc. of UKACC International Conference on Control, Coventry, UK*
  - Pfaff, R., Larkowski, T., and Burnham, K. (2011a). An application of adaptive control in the behavioural framework. In *Proc. of the 21st International Conference on Systems Engineering, Las Vegas, USA*
  - Pfaff, R., Larkowski, T., and Burnham, K. (2011b). Applying adaptive control to real-world systems in the behavioural framework. In *Proc. of the 9th European Workshop on Advanced Control and Diagnosis, Budapest, Hungary*

In the course of the research towards the problems indicated above, minor contributions to the body of knowledge were made in support of the major contributions, these are:

- Analysis of software tools: software tools capable of simulation in an acausal environment are identified and reviewed following practical requirements.

- Establishment of a link to graphical representations: the link to an appropriate graphical model representation for application in the Behavioural Framework is established and examples are given. Related publications:

  - Pfaff, R. and Burnham, K. (2008a). On abstraction and interpretability: a behavioural perspective. In *Proc. of the 19th Int. Conference on Systems Engineering, Las Vegas, USA*
  - Pfaff, R. and Burnham, K. (2009c). On abstraction and interpretability: a behavioural perspective. *Systems Science*, **35**(2), 19–26

- Analysis of nonlinear model classes: the applicability of commonly used nonlinear model classes to the Behavioural Framework is analysed and an appropriate class identified.

- Identification of material for curricula development: the material, by which the standard control engineering curriculum has to be extended to ease access to the Behavioural Framework, is identified.

- Application of behavioural techniques to practical problems: the applications presented in this thesis to practical control problems serve as examples for further applications. Selected publications:
  - Pfaff, R. and Burnham, K. (2008b). Representations of bilinear systems in the behavioural framework with application to a continuous stirred tank reactor. In *Proc. of the 6th European Workshop on Advanced Control and Diagnosis, Coventry, UK*
  - Pfaff, R. and Burnham, K. (2009b). Development of a thermal model of railway brake discs and pads for experiment design. In *Proc. of the 20th Int. Conference on Systems Engineering (ICSE), Coventry, UK*
  - Pfaff, R., Larkowski, T., and Burnham, K. (2011b). Applying adaptive control to real-world systems in the behavioural framework. In *Proc. of the 9th European Workshop on Advanced Control and Diagnosis, Budapest, Hungary*

Chapter 2

Background

> No one shall expel us from the Paradise that
> Cantor has created.
>
> David Hilbert

2.1 Introduction

This chapter in general serves to introduce the reader into the concepts that lie behind the Behavioural Framework, presents the mindset and motivation of its application and provides an initial exposure on the framework itself. With respect to the most general nature of this framework, paired with the different view of a system and its model representation, it further appears necessary to re-address some topics that are well-known and accepted in order to access the main part of this thesis free of the ballast of the classical input/output (i/o) framework.

In this sense, Section 2.2 provides some mathematical concepts, which are not frequently applied in the field of control engineering, such as set theory as well as certain concepts from linear algebra. Similar to the evolution of the mathematics curricula of engineering disciplines discussed in Appendix C, the understanding and application of the Behavioural Framework requires some new skills to be included in future curricula.

Section 2.3 explores the concept of causality from a multidisciplinary point of view, conveying an idea of causality to the reader and highlighting the pitfalls that come with this concept. The dimension of causality and the identification from observed data is a field of research in statistics and econometrics, refer to e.g. (Holland, 1986) and (Granger, 1969); and the 2011 Nobel Memorial Prize in Economic Sciences was awarded for 'empirical research on cause and effect in the macroeconomy' (Nobel Prize Committee, 2011). The relative proximity between modelling of control systems and of socio-economic systems allows a discussion of the concept of causality and a review of some of the possible misconceptions appear necessary.

In Section 2.4, modelling in the Behavioural Framework (BF)[1] is introduced. The focus of this introduction lies on modelling from physical principles, interconnections, model representations and control by interconnection. While the BF is a powerful, all encompassing framework, this section presents only the concepts of immediate use in this thesis.

[1]Acronyms are used where readability is improved, however are not used throughout the thesis.

2.2 Mathematical background

2.2.1 Preliminaries

Since the days of Georg Cantor (1845 - 1918), and especially his presentation of the non-triviality of set theory, modern mathematics is considered to be based on set theory almost in its entirety (Beutelspacher, 2001). Later, the work of Cantor was reviewed and brought into the strict axiomatic form of Zermelo-Fränkel set theory (Fraenkel *et al.*, 1973) as opposed to what is now considered as naive set theory.

Due to the fact that engineering disciplines tended to lag slightly behind novel insights in pure mathematics and the field of control engineering was branched off before the general acceptance of the set theoretic basis, the traditional field of control engineering does not make wide use of set theory nor do other engineering disciplines. The Behavioural Framework has established a close link to pure and applied mathematics by virtue of it being founded on the same set theoretic basis as modern mathematics. On the one hand, this makes it appealing from a mathematical point of view, leading to this framework being treated and extended by many mathematicians, while on the other hand, the understanding and application require new skills, currently not standard in engineering education. This situation may be considered similar to the application of Laplace transforms after the acceptance of the transfer function or statistical concepts following the introduction of estimation techniques.

Further to the material on set theory, mappings will be presented. This presentation of mappings aims to create an understanding of the concepts used in invertibility and time trajectory argumentations. Mainly for notational purposes, a brief introduction on algebraic structures such as rings, vector spaces and polynomials is provided. The section is completed by a short discussion of spaces and subspaces, having the structure of the solutions to linear differential and difference equations in mind.

The mathematical background material follows (Fischer, 2003). The formal notation of a definition will be skipped for this section, instead terms to be defined are set in italics.

2.2.2 Sets and set theory

While naive set theory is considered insufficient as a foundation for mathematics, for this thesis, the definition of a set in naive set theory suffices. A *set* is a collection of objects having a well-defined membership with the set, i.e. it can be decided for each object whether it belongs to the set.

The most simple set is a finite set

$$X = \{x_1, x_2, \dots x_n\}$$

with elements $x_i,\ i = 1 \dots n$. The fact that x_i is an element of X is denoted by $x_i \in X$. An *empty set*, i.e. a set that contains no elements, is denoted $\emptyset$. A set X' is a *subset* of X if

$$x \in X' \Rightarrow x \in X$$

for all $x \in X'$ holds, this is denoted as $X' \subseteq X$. It is common to define a subset by making use of the properties of its members

$$X' = \{x \in X | x \text{ has the property P}\}$$

Sets can be extended by the *union* of two sets, denoted

$$X \cup Y = \{z | z \in X \vee z \in Y\}$$

or a part of the elements can be excluded form the newly formed set by making use of the *intersection* operation

$$X \cap Y = \{z | z \in X \wedge z \notin Y\}$$

The *Cartesian product* of n sets yields ordered n-tuples of the form

$$X \times Y = \{(x, y) | x \in X \wedge y \in Y\}$$

Having defined these basic notions of set theory required in this thesis, it is possible to define some sets of particular interest. The set of all mappings from a set X to a set Y may be denoted by

$$Y^X = \{f | f \text{ is a function } f : X \to Y\}$$

The *power set* of a set X, denoted 2^X is considered a particular case of the set of all mappings by defining it as the set of all mappings from X to $\{0, 1\}$, with a value of 1 denoting that the mapped element is the member of the subset formed by the mapping, thus 2^X contains all subsets of X.

2.2.3 Mappings

In order to establish relations between sets, mappings are used. A *mapping* f is a relationship between two sets X and Y that assigns to each element $x \in X$ exactly one element $f(x) \in Y$. This is typically denoted as

$$f : X \to Y, \quad x \mapsto f(x)$$

While each element $x \in X$ has to be mapped to one and only one element $f(x) \in Y$, not necessarily all elements in Y have a correspondence in X. For this reason, the inverse of a mapping,

$$f^{-1} : Y \to X, f^{-1}(y) = \{x \in X | y = f(x)\}$$

in general does not fulfil the requirements of the definition of a mapping.

It is not only possible to map single elements, but also subsets of the set for which the map is defined. Assuming sets $M \subset X$ and $N \subset Y$, the *image* of a mapping, can be written as

$$f(M) = \{y \in Y | \exists x \in M \text{ with } y = f(x)\} \subseteq Y$$

The *preimage* of N under f is the set of all elements $x \in X$ that are mapped to N, in symbolic form written as

$$f^{-1}(N) = \{x \in X | f(x) \in N\} \subseteq X$$

A preimage of particular interest for mappings $f : X \to Y$ is the preimage of $0 \in Y$, the so-called *kernel* ,

$$\ker f = \{x \in X | f(x) = 0\} \subseteq X$$

Some mappings show certain properties concerning their relationship. A map $f : X \to Y$ is termed *injective* if for a given $x, x' \in X$ and $f(x) = f(x')$ always $x = x'$ follows, it is termed *surjective* if $f(X) = Y$. A mapping that is both injective and surjective is termed *bijective.* A bijective mapping can be inverted uniquely on its whole image set Y.

The *restriction* of a mapping $f : X \to Y$ to a set $I \subset X$, $f|_I$, denotes that the mapping is applied to a reduced preimage set.

2.2.4 Group, ring, field and polynomial

The algebraic structure of an *abelian group* is applied in the definitions of ring and field. An abelian group consists of a set G and a mapping

$$\star : G \times G \to G,\ (a, b) \mapsto \star(a, b)$$

having the following properties:

- $(a \star b) \star c = a \star (b \star c)$
- a neutral element $e \in G$ exists with $e \star a = a$, $\forall a \in G$ and for all $a \in G$ there exists an inverse element $a' \in G$ fulfilling $a \star a' = e$
- the operation $\star$ is commutative, i.e. $a \star b = b \star a$ for all $a, b \in G$

The concept of a ring is used in the behavioural framework for the definition of a polynomial and its properties. A *ring* is a set R together with two operations

$$\begin{aligned} + \quad &: R \times R \to R, \quad (a, b) \mapsto a + b \\ \cdot \quad &: R \times R \to R, \quad (a, b) \mapsto a \cdot b \end{aligned}$$

if R forms an abelian group with the $+$ operation, the operation $\cdot$ is associative and the distribution laws hold, i.e.

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in R$$

As an example applied in this thesis, the ring $\mathbb{Z}$, the integer numbers, may be considered.

A *field* F is a ring as above with a neutral element of addition denoted 0 where additionally the subset $F\backslash\{0\}$ forms an abelian group, i.e. in addition to the ring properties, the multiplicative inverse a^{-1} for each element $a \in F \setminus \{0\}$ is contained in F. The fields used in this thesis are the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$.

The *polynomial* f in t over a field F is an expression of the form

$$f(t) = a_0 + a_1 t + \ldots + a_n t^n$$

with the coefficients $a_0, \ldots, a_n \in F$ and a variable t. The set of all polynomials over a field F is frequently denoted by $F[t]$. The set $F[t]$ together with the canonical addition and multiplication of polynomials forms a ring. The common set of polynomials used in this thesis is $\mathbb{R}[t]$.

It is also possible to construct polynomials over the additive ring of real matrices $\mathbb{R}^{g \times q}$. The result is a matrix where each entry is a polynomial, consequently termed a *polynomial matrix*. A polynomial matrix is formalised as

$$R(t) = \begin{pmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1q}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2q}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{g1}(t) & f_{g2}(t) & \cdots & f_{gq}(t) \end{pmatrix} = A_0 + A_1 t + \ldots + A_n t^n$$

for polynomial entries

$$f_{ij}(t) = a_{0,ij} + a_{1,ij} t + \ldots + a_{n,ij} t^n$$

and matrix coefficients $A_i \in \mathbb{R}^{g \times q}$. The set of all polynomial matrices of dimensions $g \times q$ over $\mathbb{R}$ is denoted by $\mathbb{R}^{g \times q}[t]$.

2.2.5 Spaces and subspaces

The properties of vector spaces and subspaces are useful when discussing the time trajectories of a system, since the functions spaces they stem from can be considered vector spaces, as can be the behavioural equations describing them. Further, the solutions of linear and bilinear systems form subspaces or submanifolds of the signal space.

A *vector space* over a field F is a set V with an inner operation termed *addition*

$$\oplus : V \times V \to V,\ (v, w) \mapsto v + w$$

and an operation termed *scalar multiplication*

$$\cdot : F \times V \to V,\ (\lambda, w) \mapsto \lambda \cdot w$$

where the operations fulfil the following:

- V with the operation $\oplus$ forms an abelian group.
- The scalar multiplication $\cdot$ is compatible with the vector and field addition such that

$$\begin{aligned}(\lambda + \mu) \cdot v &= \lambda \cdot v \oplus \mu \cdot v \\ \lambda \cdot (v \oplus w) &= \lambda \cdot v \oplus \lambda \cdot v \\ \lambda \cdot (\mu \cdot v) &= (\lambda\mu) \cdot v \\ 1 \cdot v &= v\end{aligned}$$

  holds true for $\lambda, \mu \in F$ and $v, w \in V$.

It is common to drop the $\cdot$-notation for scalar multiplication and use the standard $+$ for the addition, since in most cases, it is obvious what is added. Further, the neutral element of V in the additive sense is frequently denoted by 0, regardless of the set it stems from. This tradition is followed for the remainder of this thesis.

A *subspace* may occur as the solution of linear equations and linear differential or difference equations. In particular, the behaviour defined by a linear differential or difference equations forms a subspace of $(\mathbb{R})^{\mathbb{R}}$. Assume V is a vector space over a field F and $S \subset V$ a subset, then the subset S is termed a subspace if the following conditions hold:

- $S \neq \emptyset$
- $v, w \in S \Rightarrow v + w \in S$
- $v \in S, \lambda \in F \Rightarrow \lambda v \in S$

The subspace axioms have the effect that for linear subspaces as a solution space for an equation, linear combinations of single solutions are also solutions. This does not apply to nonlinear systems.

2.3 Causality

2.3.1 Causality in sciences

Causality is an underlying assumption of the scientific method, which sets out in search of cause-effect relation in the physical world.

Causality or causation marks the relationship between two events, of which one is considered to be the consequence of the other. In this setting, the former is termed effect, while the latter is termed cause. Causality is known to be treated philosophically since Aristotle, however a seminal treatment that is considered highly relevant today is that of David Hume (Hume, 1740, Part III, Section XV):

1. 'The cause and effect must be contiguous in space and time.
2. The cause must be prior to the effect.
3. There must be a constant union between the cause and effect. This is what chiefly constitutes the cause-effect relation.
4. The same cause always produces the same effect, and the same effect always comes from the same cause. [...]
5. Where several different objects produce the same effect, it must be by means of some quality that we find to be common to them all. [...]
6. The difference in the effects of two resembling objects must come from a respect in which the objects are not alike. [...]
7. When an object increases or diminishes with the increase or diminution of its cause, it is to be regarded as a compounded effect, derived from the union of different effects arising from different parts of the cause. [...]
8. An object which exists for any time in its full perfection without any effect is not the sole cause of that effect, but needs to be assisted by some other force that can forward its influence and operation. [...]'

Hume further argues that causality cannot be verified by experiment and considers causality as a relation between experiences of events rather than between facts (Holland, 1986).

The (possible) causes are divided into

- necessary,
- sufficient and
- contributory

causes. Let A be the cause of effect B. If A is a necessary cause, then the occurence of B implies A. For a sufficient cause A, A implies B but B can also be caused by a second cause. A change in the contributory cause A leads to some change in the effect B, but not necessarily in all cases.

In order to inspect causality by utilising scientific experiments, assumptions are made with varying degrees of explicitness in the various scientific disciplines:

- Temporal stability: the subject or system under study is assumed to maintain its initial causal relation.
- Causal transience: the cause applied to the subject or system is assumed not to permanently change it.
- Homogeneity: when several different subjects are studied, homogeneity (possibly in a statistical sense) is assumed to be able to test the effects on different subjects.

These assumptions are made to overcome the

> '*Fundamental problem of causal interference.* It is impossible to *observe* the value of [the effect] $Y_t(u)$ and $Y_c(u)$ on the same unit [u] and, therefore, it is impossible to *observe* the effect of [the treatment] t on u.' (Holland, 1986, p. 947)

Here, in the terminology of drug testing, $Y_t(u)$ and $Y_c(u)$ denote the effect observed on a treatment and control group consisting of units u. This fundamental problem prevents the main statement expected from the scientific method to be made.

In psychology, a common result of studies is that humans tend to make *a priori* assumptions on causality of systems (Tetlock, 1985), possibly due to a focus on foreground events.

2.3.2 Causality in modelling and control

Definition

In modelling and control, frequently the causality property is assumed implicitly, e.g. by choice of a model structure, or postulated explicitly due to *a priori* assumptions on the causality of the system. A definition of a causal system commonly used in system theory relies on an implicit choice of input and output, assuming a system with input u and output y.

Definition 1 (Causal system) *A system mapping u on y is causal iff for*

$$u_1(t) = u_2(t), \forall t \leq t_0 \tag{2.1}$$

it follows that

$$y_1(t) = y_2(t), \forall t \leq t_0. \tag{2.2}$$

This definition of causality expresses Hume's second requirement, the cause must be prior to the effect, in mathematical terms and extends it to cases where cause and effect occur synchronously. It evades the identification of cause and effect in terms of the causal direction, i.e. which variable has to be considered to drive the system, since this is assumed to be known beforehand.

As an example for a system where the above definition of causality fails, consider the ideal gas law. The ideal gas law describes the relation between temperature T, volume V and pressure p of a hypothetic ideal gas and is considered a good approximation for real gases. A related technical system can be thought of as a pressure vessel, filled with an approximately ideal gas. This vessel may now be subject to heat, to change of its volume or change of the related pressure. The causal direction of this system cannot be determined uniquely.

Example 1 (Ideal Gas Law) *The relation between temperature T, volume V and pressure p for n molecules of an ideal gas is given by*

$$pV = nRT \tag{2.3}$$

where R is the gas constant, $R = 8.314 \frac{\mathrm{J}}{\mathrm{K\,mol}}$.

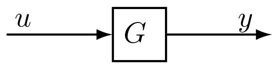


Figure 2.1: System abstracted to a signal processor

Assuming $p \in \mathcal{P}$, $T \in \mathcal{T}$, $V \in \mathcal{V}$ *and* $V \neq 0$, *it is possible to express* (2.3) *in at least three different ways as a mapping:*

$$p : (\mathcal{T} \times \mathcal{V}) \to \mathcal{P},\ p\,(T, V) = nR\tfrac{T}{V} \tag{2.4}$$

$$T : (\mathcal{P} \times \mathcal{V}) \to \mathcal{T},\ T\,(p, V) = \tfrac{pV}{nR} \tag{2.5}$$

$$V : (\mathcal{P} \times \mathcal{T}) \to \mathcal{V},\ V\,(p, T) = nR\tfrac{T}{p} \tag{2.6}$$

Indeed, all three causal directions are present in our technical world:

(2.4) *A compressor decreases the volume* V *to obtain compressed gas with an increased temperature* T *at a pressure* p.

(2.5) *An expansion cooler reduces the pressure* p *to obtain compressed gas with a decreased temperature* T *at a larger volume* V.

(2.5) *In a Stirling engine, the temperature* T *is increased in order to have pressure* p *increase the volume* V.

Causality in Modelling

In the case of modelling, the *a priori* choice of causal direction is considered to reflect the causality present in the system to be modelled, or perhaps desired for its operation. Also, as considered in further detail in Appendix C, the tools applied in the control engineering community are mostly derived from signal processing and thus, a system is reduced to a signal processor as depicted in Figure 2.1 with an arbitrary transformation mapping G.

This view of a system as a signal processor was questioned by the introduction of the Behavioural Framework some twenty-five years ago from

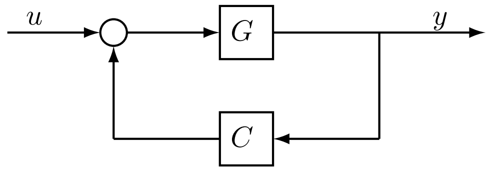


Figure 2.2: Feedback Control Scheme

a theoretical context (Zerz, 2008). This was partly inspired by the paradox introduced by the interconnection of two or more causal systems. This paradox is presented in Section 2.3.2. In recent years, following some immediately applicable techniques and tools such as Bond Graphs, an increasing amount of systems modelling is carried out in an acausal way, i.e. without *a priori* assumption of an input-output structure. This both increases the reusability of the submodels, allowing libraries of models to be accumulated, and the validity range of the overall model of the system.

Causality in Control

The predominant feedback control scheme as shown Figure 2.2 with an arbitrary plant G and controller C relies on causality of the system to be controlled, due to its concept of the output being fed back to the input. There exists a multitude of controllers that do not rely on the feedback control scheme, such as heat fins (sinks), safety valves or shock absorbers (Willems, 1997), which by design perform in making the systems behave in a desired way.

With the dominant view of control and the related tools for stability analysis, tuning and implementation of the controller being formulated in a causal framework, there is little incentive for the practising control engineer to apply acausal control techniques. However, as mentioned above, such controllers do exist but, due to their passive nature, are often not categorised as control systems, but nevertheless they manifest their advantages in a variety of applications.

Causality of Differential Equations

For the modelling task, one might consider the system to be represented by a differential equation. For lumped systems, i.e. systems that do not contain spatial coordinates, the differential equation is of the ordinary type (ODE) as opposed to partial differential equations (PDE), appearing for distributed systems, i.e. systems that comprise at least one spatial dimension. For a discussion of causality of differential equations, it is necessary to introduce the concept of continuity. For the sake of argument and simplicity, it suffices to introduce the scalar concept.

Definition 2 (Continuity (Weierstrass)) *Let f be a function $f : D \subset \mathbb{R} \to \mathbb{R}$ and a point $x \in D$. Then f is continuous in x_0 if for arbitrarily small $\epsilon > 0$ there exists a $\delta > 0$ with*

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon, \quad \forall x \in]x - \delta, x + \delta[\tag{2.7}$$

A function f is considered continuous on an interval I if (2.7) holds true for all $x_0 \in I$ with δ and ϵ as above. If a function f is n times differentiable with continuous derivative, it is common to write $f \in \mathcal{C}^n$ and $f \in \mathcal{C}^\infty$ for smooth functions.

A differential equation can be defined as an equation in which a function and its derivatives occur, as well as other variables such as time or space.

Definition 3 (Ordinary Differential Equation) *Let m and n be natural numbers and*

$$F : D \subset \mathbb{R} \times \mathbb{R}^{m(n+1)} \to \mathbb{R}^m$$

a function. An equation of the form

$$F\left(x, w, \frac{dw}{dx}, \cdots, \frac{d^n w}{dx^n}\right) = 0 \tag{2.8}$$

with an independent variable x, an $\mathbb{R}^m$-valued function F and a finite number n of derivatives of w is termed ordinary differential equation. In this context, n is the order of the differential equation and m its dimension.

The solution w of (2.8) has to be n-times differentiable in order to be a strong solution.

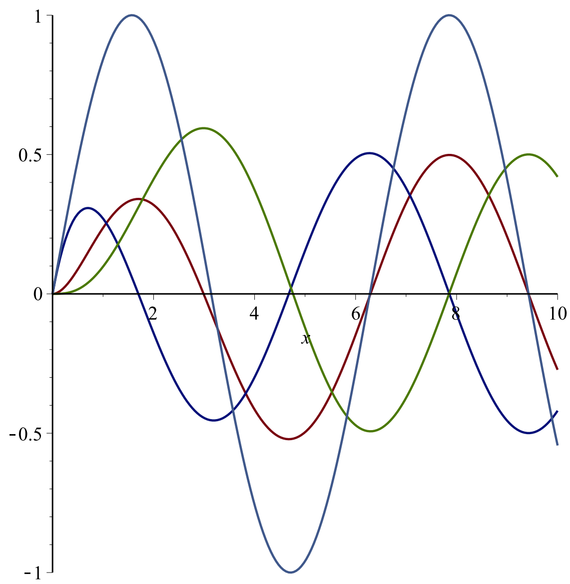


Figure 2.3: Typical solution (w, blue) for second order ODE subject to sinusoidal excitation (f, red), together with first (w', green) and second derivative (w'', purple)

Definition 4 (Strong Solution) *A function $w : D \mapsto \mathbb{R}^m$ is termed a strong solution of* (2.8) *if w is n-times differentiable and fulfils* (2.8)*.*

Since differentiability implies continuity, such a function is sufficiently smooth and especially w is continuous. This leads to a typical plot as given in Figure 2.3. Here the solution of the differential equation

$$w + 2\frac{dw}{dx} + \frac{d^2w}{dx^2} - f(x) = 0$$

with $f(x) = \sin x$ is plotted, showing the continuous disturbance f and the $\mathcal{C}^\infty$-continuous solution

$$w : x \mapsto \frac{1}{2}\exp(-x) + \frac{1}{2}\exp(-x)x - \frac{1}{2}\cos x$$

While the differential equation in (2.8) is not solved for the highest order derivative of w, a more common form of differential equation is the explicit form:

Definition 5 (Explicit Ordinary Differential Equation) *Let m and n be natural numbers and*

$$F : D \subset \mathbb{R} \times \mathbb{R}^{mn} \mapsto \mathbb{R}^m$$

a function. An equation of the form

$$\frac{d^n w}{dx^n} = F\left(x, w, \frac{dw}{dx}, \cdots, \frac{d^{n-1}w}{dx^{n-1}}\right) \tag{2.9}$$

is termed explicit ordinary differential equation.

It is generally possible to express any explicit m-dimensional n-th order differential equation as a first order differential equation of dimension $q = mn$ by utilising the following substitution:

$$\begin{aligned} \frac{dw_1}{dx} &= w_2 \\ \frac{dw_2}{dx} &= w_3 \\ &\vdots \\ \frac{dw_{n-1}}{dx} &= w_n \\ \frac{dw_n}{dx} &= F\left(x, w_1, w_2, \cdots, w_n\right) \end{aligned} \tag{2.10}$$

By making use of this property, it suffices to show the existence and uniqueness for first order differential equations of arbitrary dimension without loss of generality. The transformation of a higher order differential equation to a system of simultaneous first order differential equations is known to the control engineer as *state space* formulation.

Since the differential equation in (2.9) in general leads to multiple solutions, a further common extension of the concept is the initial condition problem. In an initial condition problem, the solution function w needs to fulfil the differential equation and one condition of the form $w(x_0) = w_0$.

Definition 6 (Initial Condition Problem) *Assume $D \subset \mathbb{R} \times \mathbb{R}^n$, $f : D \mapsto \mathbb{R}^n$ and $(x_0, w_0) \in D$. The problem is to find the function $\phi : J \mapsto \mathbb{R}^n$ defined on an interval J that solves the differential equation*

$$\frac{dw}{dx} = F(x, w) \tag{2.11}$$

on J and satisfies the initial condition

$$x_0 \in J,\ w(x_0) = w_0. \tag{2.12}$$

The existence and uniqueness of solutions to a subset with $n = 1$ of (2.9) is subject of the following theorem.

Theorem 1 (Existence and Uniqueness (Picard-Lindelöf)) *Assume $S := [x_1, x_2] \times \mathbb{R}^n$ and (x_0, w_0) a point in S. Let the mapping $F : S \to \mathbb{R}^n$ be continuous on S and satisfy the Lipschitz condition*

$$\|F(x, w_1) - F(x, w_2)\| \leq K \|w_1 - w_2\|. \tag{2.13}$$

Then there exists exactly one, on $[x_1, x_2]$ continuously differentiable, function w satisfying

$$\frac{dw}{dx} = F(x, w(x)),\ x \in [x_1, x_2] \tag{2.14}$$

$$w(x_0) = w_0. \tag{2.15}$$

Bearing in mind the time trajectories depicted in Figure 2.3, namely that the excitation signal may be discontinuous but any strong solution of the differential equation has to be sufficiently smooth to guarantee existence of the contained differentials, one might consider investigating causality on this basis. The Lipschitz condition (2.13) postulates the continuity of F and thus of the excitation signal. Considering continuity of w (implied by its differentiability) and F, it is easy to see that under the conditions of Theorem 1, a solution does not imply causality by referring to the degrees of continuity.

In control engineering however, a frequently applied solution concept that does not require the continuity of the solution function is that of weak solutions. A weak solution may be obtained even though under the conditions of Theorem 1 no solution can be guaranteed due to a lack of continuity in F. To express the concept of a weak solution, the definition of locally integrable functions and sets of zero measure is required.

Definition 7 (Locally integrable function) *A function $w : \mathbb{R} \mapsto \mathbb{R}^m$ is* *termed locally integrable if*

$$\int_a^b \|w(t)\| dt < \infty \quad \forall a, b \in \mathbb{R} \tag{2.16}$$

where $\|\cdot\|$ *is the Euclidean norm on* $\mathbb{R}$*. The space of locally integrable functions mapping* $\mathbb{R}$ *on* $\mathbb{R}^m$ *is denoted* $\mathcal{L}_1^{loc}(\mathbb{R}, \mathbb{R}^m)$ *.*

Definition 8 (Set of zero measure) *A set* $N \subset \mathbb{R}$ *has zero measure if*

$$\int_N = 0.$$

Sets of zero measure are the empty set, but also any single element subset of $\mathbb{R}$ and especially finite element sets. If a mathematical property is valid on a certain set except on a subset of zero measure, the property is said to be valid *almost everywhere*, i.e. everywhere with a finite number of exceptions.

Definition 9 (Weak solution) *Let* F *be an* n*-th order differential equation as defined in* (2.8) *and* $(\int w)(t) := \int_0^t w(\tau)d\tau$ *the integral operator. A function* $w : \mathbb{R} \mapsto \mathbb{R}^m$ *is termed weak solution if a constant vector* $c = (c_0, \cdots, c_{n-1})$ *such that* w *fulfils the integral equation*

$$F\left(x, w, \int w, \cdots, \int^n w\right) = c_0 + c_1 t \cdots, c_{n-1} t^{n-1} \tag{2.17}$$

almost everywhere.

The concept of a weak solution of a differential equation extends the space of possible solution from $\mathcal{C}^n(\mathbb{R}, \mathbb{R}^m)$ to $\mathcal{L}_1^{loc}(\mathbb{R}, \mathbb{R}^m)$. This solution concept and the associated solution space are closer to real world modelling tasks, as they allow for non-continuous solutions.

Interconnection paradox

An interesting interconnection paradox is mentioned in (Willems, 2007a). This paradox reviews the ubiquitous paradigm of modelling systems as signal processors, apart from mathematical rigour, in the context of the control engineers aim to obtain a selection of validated submodels, which can be interconnected to provide a full model.

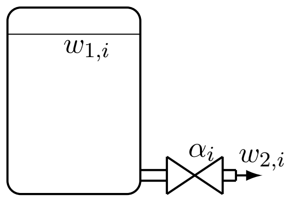


Figure 2.4: Reservoir and valve assembly

Example 2 *Assume $i \in \mathbb{N}$ fluid reservoirs with fill levels $w_{1,i}$ and surfaces A_i, controlled by a valve with opening level α_i that, under the assumption of 0 atmospheric pressure, allows a flow $w_{2,i}$ according to*

$$w_{2,i}(t) = \alpha_i(t) w_{1,i}(t) \tag{2.18}$$

where $\alpha_i \geq 0$, $\alpha_i \in \mathbb{R}$. Then these systems can be described by the differential equation

$$\frac{d}{dt} w_{1,i}(t) = -\tfrac{w_{2,i}(t)}{A_i} \tag{2.19}$$

$$w_{2,i}(t) = \alpha_i(t) w_{1,i}(t) \tag{2.20}$$

It is interesting to note that the fill level is proportional to the pressure at the bottom of the vessel, depending on the density of the fluid.

Such systems can be imagined as depicted in Figure 2.4. Viewing one system alone ($i = 1$), it seems clear that $w_{1,1}$ and α_1 are inputs and $w_{2,1}$ serves as the output of the system.

Assuming two of these reservoirs ($i = 1, 2$), connected via the pipe and two closed valves ($\alpha_i(t) = 0$, $t = 0$), which are partly opened at $t \geq 0$. At the point of interconnection, the pressures must be equal and the flows add to 0, yielding the interconnection constraints:

$$w_{1,1} = w_{2,1} \tag{2.21}$$

$$w_{2,1} + w_{2,2} = 0 \tag{2.22}$$

In this case, for one of the reservoirs, the former output w_2 becomes the input and the fill level w_1 becomes the output. As the pressure across the valve is no longer the difference between the pressure at the bottom and the

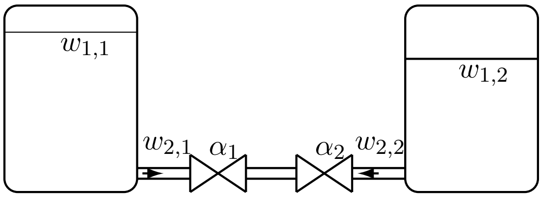


Figure 2.5: Interconnection of reservoir and valve assemblies

atmospheric pressure, the flows across the valves become

$$w_{2,1}(t) = \alpha_1(t)\,(w_{1,1}(t) - w_{1,2}(t)) \tag{2.23}$$

$$w_{2,2}(t) = \alpha_2(t)\,(w_{1,2}(t) - w_{1,1}(t)) \tag{2.24}$$

Together with the differential equations of the reservoirs

$$\frac{d}{dt}w_{1,1}(t) = -\tfrac{w_{2,1}(t)}{A_1} \tag{2.25}$$

$$\frac{d}{dt}w_{1,2}(t) = -\tfrac{w_{2,2}(t)}{A_2} \tag{2.26}$$

and the interconnection conditions (2.21) *and* (2.22)*, this yields the system equations of a system as depicted in Figure 2.5.*

In the interconnected case, the selection of input or output is far less obvious than in the case of one isolated fluid reservoir.

Example 2 serves as a counterexample against the view that the aim of reusable models is compatible with the classical view of systems as signal processors. To reach the aim of free reusability of models, regardless of the actual context, an acausal approach to modelling enables unconstrained arrangement of submodels.

2.4 Behavioural modelling

2.4.1 Introduction

The idea of a causal direction assumed *a priori* when approaching a modelling task, as well as the set of applicable methods for e.g. stability analysis developed under this assumption served as a motivation for the development

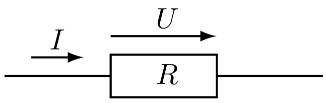


Figure 2.6: Electrical resistor circuit

of a novel framework based on a general acausality assumption.

The Behavioural Framework (BF), as developed by Willems in the early paper (Willems, 1979) and the well known tripartite paper (Willems, 1986a,b, 1987), provides a strict mathematical framework and does not require an *a priori* distinction of signals into input and output. The book (Polderman and Willems, 1998) yields a formal and precise basis, while the articles (Trentelman and Willems, 2003; Willems, 2007a) offer a concise introduction aiming towards modelling and control suitable for application to real-world systems.

While the BF is not limited in any aspect, in the literature mostly linear time invariant systems are considered, a first exposition of time varying systems is featured in (Tóth *et al.*, 2011).

The motivation for the foundation of the BF stems from the fact that most physical laws do not imply any restrictions on whether a certain variable is an input or output. Considering a resistor as in Example 3, it is impossible to say it is voltage or current driven, both variables may be input and output. The same applies to an inertial body with forces and a displacement, a more comprehensive discussion is presented in (Damic and Montgomery, 2003) and (Polderman and Willems, 1998).

Example 3 (Electrical Resistor Circuit) *Assuming a resistor of resistance R according to Ohm's law*

$$U = RI \tag{2.27}$$

with a voltage drop U along the resistor and a current I through the resistor, as depicted in Figure 2.6.

It is common to assume, as implied in (2.27)*, that this can be modelled* *appropriately by the mapping*

$$f : \mathbb{R} \to \mathbb{R}, \quad U = RI \tag{2.28}$$

Despite the fact that f is readily inverted, a causal direction is predetermined for the model, since (2.28) *is usually interpreted as 'U is a function of I'.*

When modelled following the behavioural approach, the system would be expressed as a set of value pairs (U, I) and the set

$$\{(U, I) \in \mathbb{R}^2 | \, U = RI\} \tag{2.29}$$

is the set of value pairs that may occur in a resistor circuit of resistance R. This formulation resembles more closely the observation of Georg Simon Ohm (Ohm, 1826) who claims only a proportionality between voltage and current in his experiments.

This example shows that, in addition to the theoretical advantages in approaching the modelling task in the behavioural way, it may lead to an improved insight into systems not to distinguish between input and output *a priori*, or, in the case of older discoveries, reveals the true *grandeur* of the discoverer.

From this position, it appears most natural to postulate the need for modelling techniques and related tools able to model systems without *a priori* division into input and output and thus without causality assumed before start of the modelling task. In addition to this expected gain in rigour and insight, the advent and increase in number of systems that regenerate power, such as hybrid cars or efficiency optimised machines, creates a similar need. In order to model such systems, is it necessary to model power flow inversions, i.e. inversions of causal direction, since inputs become outputs and vice versa. The common techniques, such as state space or transfer function models, are not suitable for modelling these phenomena. In order to circumvent this limitation, multiple models are used in conjunction with either blending or switching. This may lead to instabilities, as was shown in (Branicky, 1994). A framework suitable for this modelling task needs to express these power flow inversions *ab initio*, as is possible by models expressed in the BF.

2.4.2 Modelling

Polderman and Willems (Polderman and Willems, 1998) define a mathematical model as an exclusion law, that declares certain outcomes of a system as possible, while it declares others as impossible. As an example for a such an exclusion law, they use Keplers laws: Keplers laws declare that planets move on elliptic trajectories at certain angular velocities, other trajectories are impossible.

The multitude of system outcomes, possible or impossible, form the *universum* while the models selects only the possible outcomes which form the *behaviour* of the system. The question for the form of specification of this behaviour, i.e. how to narrow down the universum to the behaviour, leads to the introduction of *behavioural equations*. While behavioural equations form an effective specification of the behaviour, it is important to note that behavioural equations are nonunique. This means that a given behaviour can be expressed by a multitude of equations.

As a third important component, *variables* are considered. In the course of a modelling process, two types of variables typically occur: *manifest*, which the model aims at describing, and *latent*, which may enter the overall model in the form of internal variables necessary to express the subsystems.

The BF is different from the classical framework in that it considers the behaviour as more important than the behavioural equation, this means the starting point for the modelling task is not the equation but rather the subset of the behaviour that is possible in the system to be modelled.

The most general formal representation of a model in the BF is that of a *mathematical model*:

Definition 10 (Mathematical model) *A pair* $(\mathbb{U}, \mathcal{B})$ *consisting of a set* $\mathbb{U}$*, termed universum, and the behaviour* $\mathcal{B} \subset \mathbb{U}$ *is termed mathematical model.*

This representation makes it possible to provide the universum of outcomes (possible or not) and the subset of possible outcomes of a system to model. Since Definition 10 leaves the description of both sets $\mathbb{U}$ and $\mathcal{B}$ open, the maximum freedom remains in the modelling task. At the same time, this freedom may form part of the lacking acceptance, by offering too little guidance in the vast field of modelling.

A restriction of the very general class given by Definition 10, still general enough to cover almost any system modelling task, but more accessible for the engineer trained to model in terms of differential equations, is the class of dynamical systems.

Definition 11 (Dynamical system) *A dynamical system is a triple* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ *with* $\mathbb{T}$ *the time axis,* $\mathbb{W}$ *the signal space and* $\mathcal{B}$ *the behaviour.*

The difference between Definitions 10 and 11 is the introduction of a third element $\mathbb{T}$ and the exchange of the universum $\mathbb{U}$ in favour of $\mathbb{W}$. The added element, the time axis is a subset of $\mathbb{R}$ or $\mathbb{Z}$ for continuous or discrete time lumped models, respectively. The signal space is usually a finite dimensional vector space.

A further restriction can be found in the definition of the behaviour. While in Definition 10, the behaviour is a subset of the universum, in Definition 11 it is a subset of $\mathbb{W}^{\mathbb{T}} = \{w : \mathbb{T} \to \mathbb{W}\}$, the set of all maps from $\mathbb{T}$ to $\mathbb{W}$.

Frequently behaviours are described by differential or difference equations. In this case, $\mathcal{B}$ is the set of all mappings that fulfil the differential equations. The elements of $\mathcal{B}$ form all time trajectories admitted by the system.

An example of a dynamical system can be constructed by expressing a mass-spring-damper system in the form of Definition 11.

Example 4 *A mass-spring-damper system consisting of a mass* m*, a spring of spring rate* c *and a damper of damping coefficient* b *follows Newton's Law,*

$$m\ddot{x} = \sum F = F_e - cx - b\dot{x} \tag{2.30}$$

It is important to note that Newton's Law is acausal, i.e. it states the proportionality of the sum of all forces to the acceleration, it does not imply any causality. Consider as an example the case where the whole system is accelerated into the $-x$*-direction. In this case the body would cause a force on the source of* F_e*. Equation* (2.30) *still does not imply any causality, although it is common to consider a differential equation as causal, with the complementary function* F_e *driving the system from some initial state.*

The behavioural formulation of the above system on a continuous time-scale yields a time axis $\mathbb{T} = \mathbb{R}^+$*. The signal space contains two real variables,* *namely x and F_e, thus $\mathbb{W} = (x, F_e)^T \subseteq \mathbb{R}^2$. The behaviour is the set of all mappings from $\mathbb{T}$ to the signal space $\mathbb{W}$ that satisfy Equation* (2.30)*,*

$$\mathcal{B} = \left\{ w : \mathbb{T} \to \mathbb{W} \middle| m\ddot{x} + b\dot{x} + cx = F_e \right\} \tag{2.31}$$

The triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ then forms the representation of the dynamical system following Definition 11.

The behaviour $\mathcal{B}$ describes the set of admissible time trajectories w that are compatible with the system, in this way it forms the exclusion law postulated by Willems. Since the condition of $\mathcal{B}$ in (2.31) is defined by an ordinary differential equation, it is possible to find admissible combinations of the signals by making use of the solution schemes for differential equations.

The definition of behaviours by differential equations is frequently the case as it results from differential formulations of physical laws, but the behavioural approach to modelling is not limited to this sort of behaviours. Other ways to specify the behaviour of systems can be imagined, such as the geometric formulation in Kepler's law or the transformation between function spaces, as presented in (Pfaff *et al.*, 2006).

In the sequel of this thesis, only the dynamical systems as in Definition 11 are treated, as this model class is well accepted by practising control engineers and general enough to cover most systems.

2.4.3 Interconnections of systems

Obtaining models from first principles is typically executed not on the overall system, but in three steps. These three steps are denoted tearing, zooming and linking in (Willems, 2007a):

1. Tearing: Deconstruction of the system to obtain subsystems that can be easily modelled, white, grey or black box
2. Zooming: Modelling of the individual subsystems
3. Linking: Synthesis of the overall model from the subsystem models

The Behavioural Framework incorporates this methodology *ab initio*, especially the linking step is possible by interconnecting terminals and their associated behaviours. With submodels formulated in the input/ouput classical framework, the synthesis step may lead to inconsistencies, as shown in Example 2. These inconsistencies effectively limit the reusability of the submodels, which is of special importance for rapid prototyping applications of models.

The interconnection of two systems via their ports is formalised as follows (Polderman and Willems, 1998).

Definition 12 (Full System Interconnection) *Assuming two systems,* $\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_1)$ *and* $\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_2)$*, with the same time axis and signal space, the interconnection* $\Sigma_1 \wedge \Sigma_2$ *of these systems is defined as*

$$\Sigma_1 \wedge \Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_1 \cap \mathcal{B}_2) \tag{2.32}$$

This definition reflects that if the terminals variables are connected, their time trajectories have to be compatible with both systems. In the set theoretic approach of the BF, this means that the overall behaviour is defined by the intersection of both individual behaviours.

It is common not to connect all terminals of the subsystems, but to leave some of the variables unconnected in order to form external terminals. This is represented in Figure 2.7, where two systems Σ_1 and Σ_2 are interconnected on their common terminal w_2.

Definition 13 (System Interconnection) *With two dynamical systems* $\Sigma_1 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathcal{B}_1)$*,* $\Sigma_2 = (\mathbb{T}, \mathbb{W}_2 \times \mathbb{W}_3, \mathcal{B}_1)$ *and* $\mathbb{W}_i = w_i$*,* $1 \leq i \leq 3$*, their interconnection is defined as*

$$\Sigma_1 \wedge \Sigma_2 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \mathcal{B}) \tag{2.33}$$

with

$$\begin{aligned} \mathcal{B} = \Big\{ &(w_1, w_2, w_3)^T : \mathbb{T} \to \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3 \mid \\ &(w_1, w_2)^T \in \mathcal{B}_1 \wedge (w_2, w_3)^T \in \mathcal{B}_2 \Big\} \end{aligned} \tag{2.34}$$

Similar to the case in Definition 12, the time trajectories of the variables subject to interconnection have to be compatible with the behaviours of

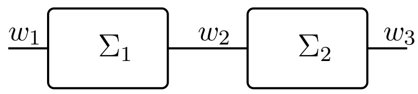


Figure 2.7: Connected subsystems

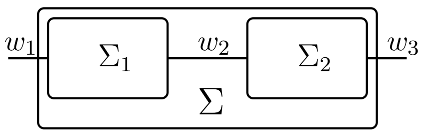


Figure 2.8: Connected subsystems with system border

both systems, i.e. these have to stem from the intersection of the systems behaviours.

While interconnections exist in the classical input/output framework, the treatment in the BF leads to a more powerful and rigorous tool especially for systems with ambiguous power or signal flow directions. This extended rigour in the approach and the set theoretic formulation of the behaviour can improve the reusability of submodels, thus a new modelling task can start with existing, validated submodels, reducing the chance of modelling errors and the effort to gain a validated model.

2.4.4 Manifest and latent variables

In the process of modelling by connecting subsystems, additional variables are introduced which express the terminals of the subsystems. The behaviour in (2.34) still contains the variable w_2, that is no longer an external variable when the overall system border is assumed as in Figure 2.8.

In the Behavioural Framework, the variables that the model aims at describing are termed *manifest variables*, while those introduced in the process of modelling the subsystems are termed *latent variables*, see (Willems, 2000).

Further important concepts connected with manifest and latent variables are those of the full and the manifest behaviour.

Since latent variables enter into almost any first principles model, they are contained in the behaviour of the interconnected system. The resulting system with latent variables is denoted $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}_\mathrm{f})$ with $\mathcal{B}_\mathrm{f} \subseteq$

$(\mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3)^{\mathbb{T}}$, according to (2.34).

Analogously it is possible to define the manifest behaviour as the behaviour of the manifest variables. In the case discussed above, w_2 is a possible latent variable, yielding the manifest behaviour

$$\begin{aligned} \mathcal{B} = \Big\{ (w_1, w_3)^T \in (\mathbb{W}_1 \times \mathbb{W}_3)^{\mathbb{T}} \,| \\ \exists w_2 \in \mathbb{W}_2^{\mathbb{T}} : (w_1, w_2, w_3)^T \in \mathcal{B}_{\text{full}} \Big\} \end{aligned} \tag{2.35}$$

The formalisation of interconnection and the classification of variables into external variables, i.e. that the system aims to describe, and internal variables, i.e. that occur while modelling of subsystems, is a further step towards reusability of submodels in the sense of the systems engineering approach.

2.4.5 Behavioural equations

Although the BF approaches the modelling task focussed on time trajectories and behaviours, i.e. solutions instead of equations, the behaviour needs specification of some kind. This specification is necessary in order to restrict the possible time trajectories from $\mathbb{W}^{\mathbb{T}}$ to those declared possible by the system. The generality of the set membership formulation makes system definitions such as

$$\begin{aligned} \Sigma &= \{\mathbb{T}, \mathbb{W}, \mathcal{B}\}, \\ \mathbb{T} &= R^+, \\ \mathbb{W} &= (F, x)^T \subseteq \mathbb{R}^2, \\ \mathcal{B} &= \Big\{ (F, x)^T : \mathbb{T} \mapsto \mathbb{W} \Big| \, \ddot{x} \text{ is proportional to } F \Big\} \end{aligned}$$

possible, however these are rarely the case. It is more common to define the behaviour with the help of equations, frequently differential or difference equations are used.

Common approaches such as differential or difference equations mostly prefer one variable over the others in terms of causality, rendering them inappropriate for an *a priori* acausal approach to modelling of systems.

To circumvent this causality assumed *a priori*, the notation of *Behavioural Equations* is introduced (Willems, 1997).

Definition 14 (Behavioural Equations) *For a universum $\mathbb{U}$ and a set $\mathbb{E}$, be $f_1,\ f_2 : \mathbb{U} \to \mathbb{E}$. A mathematical model $(\mathbb{U}, \mathcal{B})$ is said to be defined by behavioural equations if $\mathcal{B} = \{u \in \mathbb{U} | f_1(u) = f_2(u)\}$.*

While this definition is not used in the remainder of this thesis, it illustrates an important concept of equations in the BF. Often in the description of laws of nature or society, equilibria instead of assignments form the basis of these laws, as an example Newton's law, Kirchhoff's current law or the law of supply and demand may be considered.

An equilibrium is different to an assignment in that it possible to vary the value of any of the variables used to express the equilibrium and to find new equilibrium values for the others. However the common interpretation of the equality expressing Newton's law is that forces applied to a mass lead to an acceleration. While this holds true, also accelerations result in forces.

A special form of Definition 14, suited for linear dynamical systems is of more importance for this thesis, the kernel representation for behavioural systems proposed in (Polderman and Willems, 1998).

Definition 15 (Kernel representation) *With a time series of variables of a system $\mathbf{w}(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots, w_q(\cdot))^T$ and $R_0, R_1, \ldots, R_l \in \mathbb{R}^{g \times q}$, a kernel representation is given by*

$$R_l \Delta^l \mathbf{w} + R_{l-1} \Delta^{l-1} \mathbf{w} + \cdots + R_0 \mathbf{w} = 0 \tag{2.36}$$

where Δ denotes the differential or the shift operator for continuous or discrete time systems, respectively.

This equation can then be used to formally define a dynamical system with $\mathbb{T} \subseteq \mathbb{R}$ or $\mathbb{T} \subseteq \mathbb{Z}$, $\mathbb{W} \subseteq \mathbb{R}^q$ and the behaviour

$$\begin{aligned} \mathcal{B}(R) = \Big\{ & w : T \mapsto \mathbb{R}^q \\ & | R_l \Delta^l \mathbf{w} + R_{l-1} \Delta^{l-1} \mathbf{w} + \cdots + R_0 \mathbf{w} = 0 \, \forall t \in \mathbb{T} \Big\} \end{aligned} \tag{2.37}$$

It is possible to write (2.36) in a more convenient form by use of the polynomial matrix $R \in \mathbb{R}^{g \times q}[s]$, given by

$$R(s) = R_l s^l + R_{l-1} s^{l-1} + \cdots + R_0$$

and the appropriate operator Δ

$$R(\Delta)\mathbf{w} = \mathbf{0} \tag{2.38}$$

The representation in (2.38) is termed kernel representation due to its close relation to the kernel of a matrix or linear mapping. The kernel forms a linear subset of the image of the mapping, thus for a vector space $\mathbb{W}$ the behaviour $\mathcal{B}(R)$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$.

2.5 Concluding remarks

Starting with a review of some mathematical concepts, not commonly applied in control engineering, this chapter aims to introduce the reader to the Behavioural Framework and the motivation for its development and application. Among these mathematical concepts, set theory and mappings play a vital role in the BF, since the difference of the Behavioural Framework and the classical i/o framework, beside the avoidance of *a priori* assumptions on the input/output structure, lies in the definition of systems in terms of time trajectories, i.e. mappings from time axis to signal space, and the behaviour, i.e. the set of all time trajectories declared possible by the system.

Reflecting on the attractiveness of the BF from both the practising control engineers and the applied mathematicians point of view, it can be said that while the work with sets and mappings instead of Laplace or z-Transforms from the very beginning appears cumbersome, it brings the domain of systems and control closer to he current state of mathematics, which is based on set theory.

When modelling in the i/o framework, it is necessary to assume an input/output structure from the beginning of the modelling task. This, together with the commonly used model structures, inevitably leads to assumptions on which signals of a system are cause and which are effect. In this way, the main output assumed from scientific work, namely the identification of cause and effect, is evaded. The concept of causality is thus reviewed in this chapter and the special situation of differential equations and causality is discussed.

The historical evaluation of the field of systems and control gives indication that it was initiated free of *a priori* assumptions on the causal direction of the signals. Plenty of successful and sophisticated applications as well as far-looking theoretical developments were made in an acausal framework until, decades after 'On Governors', systems were considered as signal processors. The systems were started to be viewed as signal processors mainly for two reasons, one being the availability of tools in the telecommunications domain, the other being the inclusion of related mathematical techniques in the engineering curricula, at least for specialised disciplines.

Initially, this transfer of existing mathematical tools lead to a shift towards theory in systems and control, while the majority of practising engineers in the field did not master the tools to follow the theoretical discussions of their colleagues. This movement was termed 'The Gap' and it may be feared that, due to the relative inaccessibility of the Behavioural Framework, the same may occur, for almost the same reasons. While the BF brings modelling and control closer to mathematics and enables the field to follow and profit from the developments in mathematics similarly to physics, it has a tendency to keep the control engineers at a distance due to its notation and abstract concepts.

The chapter is finalised by an introduction to the Behavioural Framework, that despite its brevity introduces the main concepts required in this thesis and thus is sufficient also for an application of the BF to practical control engineering problems. In this sense, this chapter not only serves as a revision of material from a point of view related to the Behavioural Framework, but also groups the material necessary for an introductory course on the BF. Such lectures within the Masters level curricula would ease access to further works in the BF significantly and may therefore increase its popularity.

Chapter 3

Related Work

> If I have seen further it is by standing on ye sholders of Giants.
>
> Isaac Newton

3.1 Introduction

This chapter provides a literature survey of relevant publications in the two fields under application in this thesis, nonlinear systems and the Behavioural Framework. The dominating first section gives an outline of work in the Behavioural Framework, ranging form the foundational papers to recent developments in time varying behaviours, touching also the theoretical side approached by applied mathematicians.

As a second section, an overview over nonlinear model classes from a practising control engineers point of view is given. It presents relevant model classes and introduces criteria for model structure selection. The topic of model selection is important in that it is a key decision in modelling of nonlinear systems.

A third section presents some simulation tools for acausal modelling and simulation. These tools are reviewed both from the perspective of application in the development process of a company and the requirements created by the desired application of the BF.

Finally, a fourth section puts an interdisciplinary view on the topic of causality. Causality and modelling without *a priori* assumptions on the causal direction of the signal flow is also of interest in other disciplines, these include Medicine, Econometrics and Sociology. Briefly addressing these topics makes it possible to view this thesis in a wider context and consider control by interconnection also as a stabilising element e.g. in a monetary system.

3.2 Work in the Behavioural Framework

3.2.1 Foundation of the Behavioural Framework

The Behavioural Framework is founded on the article (Willems, 1979), however a more popular paper on this subject is the well-known series of three papers 'From time series to linear systems', (Willems, 1986a), (Willems, 1986b) and (Willems, 1987). While the older article receives 114 citations according to google scholar, the first part of the tripartitie paper gathers 393 citations, also according to google scholar. Of these almost 400 citations, since 2008 41 were made, indicating that the topic appears to be still interesting for the scientific community.

The first part of the series treats linear time invariant systems and especially puts *a priori* distinguishing the signals of system into input and output in question. The argument, as for the whole BF, is based on the system definition as Definition 11, with the paper being limited in scope to the discrete time case. In order to define the behaviour $\mathcal{B}$ of the systems considered, several structures for the behavioural equations are introduced:

(AR) The autoregressive model or kernel representation as introduced in Section 2.4.5, $R(\triangle)w = 0$.[1]

(AUX) A system incorporating auxiliary variables ξ, without an input/output structure assumed, $R'(\triangle)w = R''(\triangle)\xi$.

(i/o) A model with predefined input/output structure and a proper transfer function, $P(\triangle)y = Q(\triangle)u$, $w = (u, y)^T$.

(S) A state space system $\triangle x = A'x + B'u,\ w = C'x + D'v$

(i/s/o) The input-state-output system $\triangle x = Ax + Bu$, $w = Cx + Du$, $w = (u, y)^T$

The auxiliary variables of the (AUX) structure frequently occur while modelling by tearing, zooming and linking, i.e. by modelling of subsystems and linking these to form an overall model. Being of the same equilibrium based representation as the (AR) model, it implies no causality in terms of an assignment of an output according to an input. The proper transfer function of the (i/o) model as such does not imply causality, hence this model structure with a biproper, i.e. proper with proper inverse, transfer function is acausal. The *a priori* distinction of input and output however makes the model structure little applicable within the BF. The model class (S) is causal in that the variable u can only define the state x after one time step delay or only via its derivative, for discrete and continuous time system respectively. In the (i/s/o) model structure, no causality beside the distinction between input and output signals is assumed beforehand, since the u and y can influence mutually via the matrix D.

[1]The terminology autoregressive model is not applicable to continuous time systems, while the term kernel representation is suitable for both cases, therefore the latter will be used for the remainder of this thesis except for cases that do not apply to continuous time models.

Further to the comparison and partly introduction of these model structures, three theorems are presented that guarantee the ability to transfer between the individual model structures by help of transformation matrices. For the transformation of the state space system to an (AR) representation exhibiting the same behaviour, an algorithm is provided.

The introduction of model structures expressing behaviours without *a priori* assumptions on their input/output structure is the most commonly applied contribution of this paper, while the second is of slightly more theoretical nature.

The qualitative properties of such models, in particular linearity, time invariance and completeness is treated in detail and considered the main result by the author. A dynamical system $\Sigma = \{\mathbb{T}, \mathbb{W}, \mathcal{B}\}$ is called *linear* if $\mathbb{W}$ is a vector space and $\mathcal{B}$ forms a linear subspace of $\mathbb{W}^{\mathbb{T}}$. The system is *time invariant* if $\mathbb{T}$ is an additive semigroup in $\mathbb{R}$, i.e. for $t_1, t_2 \in \mathbb{T}$ follows $t_1 + t_2 \in \mathbb{T}$, and if for $t \in \mathbb{T}$, the shifted behaviour is a subset of the unshifted, $\sigma^t \mathcal{B} \subset \mathcal{B}$. In this context, σ denotes the discrete time backward shift operator satisfying $\left(\sigma^t f\right)(\tau) = f(\tau + t)$.

Further to these standard requirements presented in the notation of the BF, a new property is introduced. This property is completeness:

Definition 16 (Complete system) *A system $\Sigma = \{\mathbb{T}, \mathbb{W}, \mathcal{B}\}$ is complete if*

$$w \in \mathcal{B} \Leftrightarrow w|_{\mathbb{T} \cap [t_o, t_1]} \in \mathcal{B}|_{\mathbb{T} \cap [t_o, t_1]}$$

for all $-\infty < t_0 \leq t_1 < \infty$.

From the property of completeness, an existence theorem for discrete time systems is derived:

Theorem 2 (Existence of (AR) representation) *Be $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}^+$ and $\Sigma = \{\mathbb{T}, \mathbb{R}^q, \mathcal{B}\}$. Then there exists a polynomial matrix R with $\mathcal{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{T}} \,|R(\sigma) w = 0 \right\}$ iff Σ is linear, time invariant and complete, i.e. iff $\mathcal{B}$ is linear, shift invariant and closed in the topology of pointwise convergence.*

Since the existence of an (AR) representation implies finite dimensionality, this property no longer has to be postulated for a linear time invariant system. Instead of postulating this property, it follows from the reasonable assumption that $\mathcal{B}$ is closed in the topology of pointwise convergence, i.e. the pointwise limit values of series of functions lie in B.

The second part of the tripartite paper is dedicated to exact modelling, as opposed to approximate modelling under the assumption of mismatch between model and data. In the argument of Willems, it is logical to handle exact modelling first and later address the approximate modelling problem. In the existing work on system identification, this logical order is inversed.

The typical approach to the inverse problem of finding a model that fits a given time series, in an approximate or exact sense, is to select a model structure first and then choose the variables to optimise the fit. Since these models are usually falsified by the data, i.e. the data contains at least one point that the model cannot explain, a randomized element, the error term, is introduced.

In this sense, the paper sets out to model a given time series with a powerful model that is unfalsified by the data, i.e. it explains all data points exactly. The notion of power of a model stems from a Popperian falsifiability point of view (Popper, 1963). Falsifiability assumes that it is impossible to deduce a model from a finite data set, instead a model is assumed arbitrarily and tested on the experimental data. If the model is able to explain all the data, the model is assumed to be correct, hence a model that allows less is considered more powerful than a model that allows almost any data set.

For the purpose of a review of the modelling and identification technique without conceptions of existing solutions, a terminology is introduced, this terminology will be presented in Section 6.2.3 of this thesis.

While the general task of defining a set that contains the measured data and in this way is not falsified by the data, is a simple task, it is far more complex to derive the equivalent most powerful (AR) relation. This model, which is the most powerful and unfalsified one is termed most powerful unfalsified model (MPUM). A number of algorithms is presented, not only solving this task but also supplying different model structures and minimal representations.

Concluding this part of the series, Willems claims that the algorithms presented in this part, i.e. those for exact modelling, will eventually be of interest in identification, adaptive signal processing and adaptive control algorithms. Notwithstanding the sensibility of the exact modelling approach from an applied mathematics perspective, the adaptive techniques for control rely to a large extent on the *a priori* selection of the model structure, as not to switch controller structure based on the structure selection of the identification part. Since this is the fundamental difference in this approach, the success of the exact modelling approach in adaptive control does not appear very likely in the current state of control and also taking into account the developments in control in the BF.

The third part of the tripartite paper deals with the more practical task of fitting models approximately to a set of data. While it is possible to generalise the exact modelling algorithms presented in the second part to approximate modelling, Willems considers it more logical to include approximate fitting in the problem formulation.

The approach chosen to find an optimal approximate model differs from the common statistical framework in that either low complexity or high accuracy is strived for, instead of qualities such as unbiasedness or consistency. Consequently, the two dimensions to assess a model will be the complexity of the model and the misfit between data and model. The complexity can be thought of as being the inverse of the model power, the misfit is the degree by which the model missed the data. A more detailed introduction into the concepts and termini is subject of Section 6.2.3, here also the algorithms are outlined.

In order to proceed in the proposed direction, two methodologies are presented, allowing to model with limited complexity or with limited misfit. Algorithms are developed to search for the optimal model under limited complexity or misfit and the correctness of the algorithms is proven mathematically.

Two application examples are presented, the first of the examples identifies an optimal approximate model with limited misfit of an autonomous system without noise considered; the same simulation is repeated with added noise on both signals and a saturation type nonlinearity forming a Wiener system. Both parts of the experiment show appropriate performance of the algorithm. The second experiment aims to find an optimal approximate model, also with limiting the misfit, for an acausal impulse response. Here a potential shortcoming of the algorithms becomes apparent from the inspection of the very slowly decreasing misfit vector. Both algorithms in theory terminate, which is a proven property of the algorithm, however an approximate model of the specified complexity or misfit does not necessarily exist and if, it may be a very sensitive model. This shortcoming is circumvented by using heuristics that resembles more the practical engineering approach than strictness. A different approach, already proposed in the third part of Willems paper, is not followed. Such an approach is developed in Chapter 6.

3.2.2 Further development of the Behavioural Framework

Formalisation of system interconnection

Based on the foundations laid in the tripartite paper, several strands for further development were followed. One of these is the integration of a system interconnection methodology into the BF (Willems, 2008b). As a motivation for this work serves the fact that systems, physical or man-made, usually consist of a number of interconnected and interacting subsystems. The aim is to be able to model these systems by identifying the subsystems and modelling these separately. In this respect, the input/output framework is more restrictive than one believes, as shown in Example 2.

To overcome these limitations, Willems proposes an approach in the BF that consists of three stages:

1. Tearing: view the overall system as interconnection of subsystems.
2. Zooming: derive models for the subsystems.
3. Linking: modelling of the interconnections and synthesis of the overall system.

In the BF, the third step refers to variable sharing rather than output-input connection, this reduces the amount of *a priori* assumptions that have to be made in the first two stages of the process.

This approach, as the Behavioural Framework, is considered to treat a model for what it is, as an exclusion law in the Popperian sense. The methodology of Tearing, Zooming and Linking is claimed to form an excellent illustration of the appropriateness of the BF, despite the observation that the concept of interacting terminals may not be right for certain interactions, such as actions at a distance or friction changing between rolling and sliding.

The concept of Tearing, Linking and Zooming is further formalised in the article, based on five concepts:

Type of terminal	Variables	Universum
electrical	(voltage, current)	$\mathbb{R}^2$
1-D mechanic	(force, position)	$\mathbb{R}^2$
2-D mechanical	(force, position, torque, angle)	$\mathbb{R}^3 \times [0, 2\pi)$
thermal	(temperature, heat-flow)	$\mathbb{R}^+ \times \mathbb{R}$
fluidic	(pressure, mass-flow)	$\mathbb{R}^2$

Table 3.1: Terminal types, variables and their universum

1. Terminals
2. Modules, that may be parametrised
3. Interconnection architecture
4. Module embedding
5. Manifest variable assignment

A terminal is defined by its type, defining a universum of terminal variables, measuring physical quantities of the interaction of this terminal with the environment. A list of terminal type examples (a part of (Willems, 2008b, Tab. 1)) is given in Table 3.1.

A module is specified by a type and a behaviour, the type defines the kind of a system, e.g. mechanical, electrical, while the behaviour defines the behaviour of the module variables. An interconnection architecture yields a representation of the interconnections of the single terminals, as a form, the undirected graphs are proposed. The module embedding serves as a definition of the arrangement of the subsystems. The interconnection and the arrangement requires a definition of the interconnection equations, forming a model of which laws govern the interconnection. A number of interconnection equations is shown in Table 3.2, representing a part of (Willems, 2008b, Tab. 4). Finally, the manifest variable assignment defines which variables interact with the environment and which were added in the process of modelling by tearing, linking and zooming.

The methodology has the advantage of being systematic and leading to modular, re-usable, extendable models. This is achieved at the cost of handling a high number of variables and uncommon concepts for the system engineer. It can be compared to electrical systems theory and bond graphs,

Pair of terminals	Variables T1	Variables T2	Interconnection equation
electrical	(V_1, I_1)	(V_2, I_2)	$V_1 = V_2,\ I_1 + I_2 = 0$
1-D mechanic	(F_1, x_1)	(F_2, x_2)	$x_1 = x_2,\ F_1 + F_2 = 0$
thermal	(Q_1, T_1)	(Q_2, T_2)	$T_1 = T_2,\ Q_1 + Q_2 = 0$
fluidic	(p_1, f_1)	(p_2, f_2)	$p_1 = p_2,\ f_1 + f_2 = 0$

Table 3.2: Interconnection laws

from the former it exhibits the difference that the subsystems are in the leaves, not the branches of the tree, while from the latter it is different by not handling energy in the subsystems.

Control in the Behavioural Framework

Having established the systematic approach to modelling by interconnecting subsystems, a next strand to follow is that of control in the BF. While the general concept of control in the BF is presented in Section 7.2.1, here a focus will be put on the behavioural versions of pole placement and stabilization, following (Praagman *et al.*, 2007). These versions differ from the standard versions since the BF distinguishes between a behaviour $\mathcal{B}$ and its representation in the form of equations.

A system $\Sigma = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}\}$ is termed *controllable* iff it admits an image representation

$$\mathcal{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \,|\exists l \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) :\ w = Ml\}$$

This is denoted as $\mathcal{B} = \mathrm{im}(M)$ and the kernel representation as $\mathcal{B} = \ker(R)$. A system with $\mathcal{B} = \ker(R)$ is controllable if and only if $\mathrm{rank}\,(R(\lambda))$ is independent of λ for $\lambda \in \mathbb{C}$ and it is *stabilisable* iff $\mathrm{rank}\,(R(\lambda))$ is independent of λ for $\lambda \in \mathbb{C}^+$. The set of all linear differential systems of dimension q is denoted $\mathfrak{L}^q$.

The behavioural version of observability assumes a linear system with system variable $w = (w_1, w_2)$. The variable w_2 is termed *detectable* from w_1 if $(w_1, w_2), (w_1, w_2') \in \mathcal{B}$ implies $\lim_{t\to\infty}(w_w(t) - w_2'(t)) = 0$. For a representation of the form $R_1 w + R_2 w = 0$, w_2 is *observable* from w_1 iff $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$.

A concept of great importance in the BF and in the sequel is that of *elimination*. For a behaviour $\mathcal{B} = \ker(R_1, R_2)$, a representation of $\mathcal{B}_{w1}$ can be obtained by applying an unimodular matrix V with $VR_2 = \mathrm{col}\,(R_{12}, 0)$

and R_{12} having full row rank. Then a partition $VR_1 = \mathrm{col}\,(R_{11}, R_{21})$ yields $\mathcal{B}_{w1} = \ker(R_{21})$.

Assuming a plant with behaviour $\mathcal{P}_{\text{full}}$ and system variables (w, c), a controller with behaviour $\mathcal{C}$ and system variable c and their interconnection

$$\mathcal{K}_{\text{full}} = \mathcal{P}_{\text{full}} \wedge \mathcal{C} = \{(w,c)|(w,c) \in \mathcal{P}_{\text{full}} \wedge c \in \mathcal{C}\}$$

termed the *full controlled behaviour*. This interconnection is termed regular if the sum of the output cardinalities of controller and plant is equal to that of the full controlled behaviour, i.e. $\mathtt{p}\,(\mathcal{K}_{\text{full}}\,(\mathcal{C})) = \mathtt{p}\,(\mathcal{P}_{\text{full}}) + \mathtt{p}\,(\mathcal{C})$. The output cardinality $\mathtt{p}(\mathcal{B})$ for a system with behaviour

$$\mathcal{B} = \left\{(w,l) : \mathbb{T} \to \mathbb{R}^{q+k} | Rw = Ml\right\}$$

is calculated as $\mathtt{p}(\mathcal{B}) = \mathrm{rank}\,(R, M) - \mathrm{rank}\,(M)$. A regular interconnection of a controller to a plant does not impose any laws on the plant that it already fulfils.

The above properties of a linear differential system allow for the following

Theorem 3 (Existence of a stabilising controller) *Be $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ a plant with signals $w \in \mathbb{R}^q$ and $c \in \mathbb{R}^k$. There exists a stabilising controller $\mathcal{C}$ iff $(\mathcal{P}_{\text{full}})_w$ is stabilisable and in $\mathcal{P}_{\text{full}}$ the variables w are detectable from c.*

On the basis built above, the two main problems are addressed, the first is the parametrisation of a controller for a defined plant and overall behaviour by regular interconnection, the second is the parametrisation of the stabilising controller for a given plant. In the development of the solution, first the full interconnection case, i.e. the case where no external variables remain after interconnection, is studied. This case, due to no external variables, has limited practical applicability. The case of partial interconnection is developed starting with the assumption that c is observable from w.

The definition of *manifest plant behaviour* is introduced, denoted $(\mathcal{P}_{\text{full}})_w$, that is obtained from $\mathcal{P}_{\text{full}}$ by eliminating c as well as the *hidden behaviour* $\mathcal{N} = \{w|(w,0) \in \mathcal{P}_{\text{full}}\}$. It is able to state conditions for implementability and regular implementability according to the following

Theorem 4 (Implementability (Praagman *et al.*, 2007)) *A controlled behaviour $\mathcal{K} \in \mathfrak{L}^q$ is implementable by partial interconnection through c with* *respect to* $\mathcal{P}_{\text{full}}$ *iff* $\mathcal{N} \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$. *A controlled behaviour* $\mathcal{K} \in \mathfrak{L}^q$ *is regularly implementable by partial interconnection through c with respect to* $\mathcal{P}_{\text{full}}$ *iff* $\mathcal{N} \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$ *and* $\mathcal{K}$ *is regularly implementable with respect to* $(\mathcal{P}_{\text{full}})$ *by full interconnection.*

The main result for the observable case with partial interconnection is the parametrisation of all controllers that implement a given overall behaviour $\mathcal{K}$ with respect to the full plant behaviour $\mathcal{P}_{\text{full}}$. This is achieved by calculation of representations of $(\mathcal{P}_{\text{full}})_c$ and $(\mathcal{L}_{\text{full}}(K))_c$, the controlled variable of the subset of $\mathcal{P}_{\text{full}}$ that shows $w \in \mathcal{K}$, i.e. $\mathcal{L}_{\text{full}}(K) = \{(w,c) \in \mathcal{P}_{\text{full}} | w \in \mathcal{K}\}$.

Theorem 5 (Controller Parameterisation (Praagman *et al.*, 2007)) *Let* $\mathcal{P}_{\text{full}}$ *be* $\in \mathfrak{L}^{q+k}$ *with* (w,c) *the system variable, c observable from w and* $\mathcal{P}_{\text{full}} = \ker(R_1, R_2)$ *a minimal representation. Assume* $\mathcal{K} \in \mathfrak{L}^q$ *regularly implementable through c with respect to* $\mathcal{P}_{\text{full}}$ *and* $\mathcal{K} = \ker(R)$ *the associated minimal representation. The parametrisation of all controllers is achieved by finding polynomial matrices* V_1*,* V_2*,* F_1 *and* W *according to following algorithm: Choose matrices*

- V_2 *such that* $\operatorname{im}(R_1) = \ker(V_2)$
- V_1 *such that col* (V_1, V_2) *is unimodular*
- M *such that* $\operatorname{im}(M) = (\ker(V_2 R_2))_{\text{cont}}$ *with* $M(\lambda)$ *having full column rank for all* λ
- F_1 *such that* $K = F_1 V_1 R_1$
- Q *such that* $\operatorname{im}(F_1 V_1 R_2 M) = \ker(Q)$ *with* Q *having full row rank*
- W *such that* $\operatorname{col}(Q, W)$ *is unimodular*

For all $\mathcal{C} \in \mathfrak{L}^k$ *with* $\mathcal{C} = \ker(C)$*, the following statements are equivalent:*

- *The controller* $\mathcal{C}$ *regularly implements* $\mathcal{K}$ *through c with respect to* $\mathcal{P}_{\text{full}}$*,* $\mathcal{C} = \ker(C)$ *is a minimal representation.*
- *There exist* $G \in \mathbb{R}[s]^{l \times m}$ *and a unimodular matrix* U *with*

$$C = (UWF_1V_1 + GV_2)\, R_2$$

The nonobservable case is reduced to the observable case by modification of the system matrix R_2 representing the same behaviour.

This work on control in the BF is of great value in applied mathematics due to its proof of existence of a controller for a given behaviour for a linear differential system, especially since this proof is lead in a constructive way. From a control applications point of view, the parametrisation of a controller via a number of matrix equations appears cumbersome and likely a more heuristic approach will be chosen.

The problem of controller synthesis can also be viewed as interconnecting a controller such that the overall system becomes dissipative. Such an approach is chosen in (Willems and Trentelman, 2002) and the results given above are obtained. In the second part of the paper (Trentelman and Willems, 2002), setups for disturbance attenuation or passivation as control targets are presented and the cases of feedback control and filtering are addressed.

A refinement of the dissipativity property of a controller is presented in (Rapisarda and Kojima, 2010), where the equivalence of the controller imposing dissipation and the stabilization by full interconnection is shown.

Adaptive control in the Behavioural Framework

In order to be able to achieve behavioural control also for time varying systems, it is necessary to extend the control paradigm to adaptive control. Results of this extension are given in (Polderman, 2000). The aim of this article is to provide a controller achieving a specified controlled behaviour by regular interconnection of an appropriate controller.

The approach to this problem is to use sampled data to find the most powerful unfalsified model at a given time instant and to find corresponding additional constraints such that they can be imposed by a regularly connected controller.

For the development of this approach, the definition of the manifest variable w and the latent control variable is extended, components of w can be components of c and vice versa. Further, in addition to the concept of a regular interconnection, the *regular feedback* interconnection is used. An interconnection is termed a regular feedback interconnection if it is regular and the McMillan degree of $\operatorname{col}(R_1, R_2)$ is equal to the sum of McMillan degrees of R_1 and R_2.

An alternative way to check whether the desired behaviour can be achieved is by considering two extreme behaviours, the uncontrolled and the maximally controlled behaviour, $\mathcal{B}_{\text{unc}}$ and $\mathcal{B}_{\max}$, respectively.

$$\mathcal{B}_{\text{unc}} = \left\{(w,c) \in \mathcal{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{q\times d}\right) | R\left(\tfrac{d}{dt}\right) w = M\left(\tfrac{d}{dt}\right) c\right\} \tag{3.1}$$

$$\mathcal{B}_{\max} = \left\{(w,c) \in \mathcal{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{q\times d}\right) | R\left(\tfrac{d}{dt}\right) w = 0 \ c = 0\right\} \tag{3.2}$$

The maximally controlled behaviour resembles the hidden behaviour $\mathcal{N}$ applied by (Praagman *et al.*, 2007) with exception of the common usage of the components in w and c.

The existence of a controller for a given behaviour is given by the following theorem (Polderman, 2000, Th. 2.1).

Theorem 6 *Let*

$$\mathcal{C} \subset \mathcal{B}_{\text{unc}} = \left\{(w,c) \in \mathcal{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{q\times d}\right) | \tilde{R}\left(\frac{d}{dt}\right) w = \tilde{M}\left(\frac{d}{dt}\right) c\right\} \tag{3.3}$$

for polynomial matrices $\tilde{R}$ and $\tilde{M}$. There exists a polynomial matrix C such that $\mathcal{C}$ is given by (3.3) *iff $\mathcal{B}_{\max} \subset \mathcal{C}$.*

Adaptive control is approached by admitting R and M to be unknown and time varying, however satisfying $\mathcal{B}_{\max} \subset \mathcal{C}_{\text{des}}$. An iterative scheme is proposed in which the evolution of the variables (w,c) is observed on an interval I_0 and the behaviour is modelled using an extension of the MPUM following (Willems, 1986b). Based on this model of the behaviour, the controller is derived. The iterative scheme is proven to converge to the desired behaviour after a finite number of iterations.

The approach followed by the author is appropriate under the assumptions posed, which includes the ability to observe noise-free variables of both plant and controller. The assumption of a noise-free measurement is not valid for real-world systems due to noise effects and quantisation in the sensors. Further, the resulting controller is not subject to any structural limitations, which does not guarantee its technical implementation in a real-world system. This makes the paper, although coming closer to the requirements of the practising control engineer, appear motivated by rather theoretical reasoning.

Another publication related to the topic of adaptive control in the BF, from a perspective less close to application is (Ilchmann and Mehrmann, 2006). In this publication, time-varying differential systems are considered and results, such as controllability, observability and autonomy, originating from the linear time-invariant paradigm are generalised to time-varying differential-algebraic systems. These results will be presented below in the case of real numbers, as this case is closer to applications.

The differential polynomials applied in a time varying setup do not stem from a polynomial ring $\mathbb{R}^{g\times q}[s]$, but rather have their polynomial parameters in $\mathcal{A}$, the ring of real analytic functions $f : \mathbb{R} \to \mathbb{R}$. The ring $\mathcal{A}$ can be applied to construct a skew polynomial ring $\mathcal{A}[s]$, which can be arranged in matrix shape to form a skew polynomial ring $\mathcal{A}[s]^{g\times q}$. The behaviour of the respective differential system in kernel representation is

$$\ker R = \left\{ w \in \mathcal{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^q\right) | R\left(\frac{d}{dt}\right) w(\tau) = 0 \text{ for almost all } \tau \in \mathbb{R} \right\}$$

where $R(s) \in \mathcal{A}[s]^{g\times q}$.

The generalisation of the concept of controllability for a linear time-varying (LTV) system is subject of the following theorem (Ilchmann and Mehrmann, 2006, Th. 3.2), of which a version for real parameters is presented.

Theorem 7 (Controllability for LTV systems) *Assume $R(s) \in \mathcal{A}[s]^{g\times q}$ has full row rank. Then the behaviour $\ker R$ is controllable almost everywhere if and only if $R(s)$ is right invertible.*

The proof employs the Teichmüller-Nakayama normal form and shows that indeed a set of zero measure remains that is not controllable.

Another concept generalised to LTV systems is that of controllability. In Belur and Trentelman (2002), similar properties of the system matrices guaranteed controllability and observability, the theorem (Ilchmann and Mehrmann, 2006, Th. 5.6) resembles this result, stated below in a version for real parameters.

Theorem 8 (Observability for LTV systems) *For $R(s) \in \mathcal{A}[s]^{g\times q}$ the two statements are equivalent*

1. *The behaviour $\ker R$ is locally controllable almost everywhere.*
2. *The variable l is locally observable almost everywhere from w with*

respect to the behaviour defined by

$$\left[Id_q, R^{ad}\right]\begin{pmatrix} w \\ l \end{pmatrix}$$

where R^{ad} denotes the adjoint of R, defined as

$$\cdot^{ad} : \mathcal{A}[s]^{g\times q} \to \mathcal{A}[s]^{g\times q},\ \sum_{i=0}^{k} P_i s^i \mapsto \left(\sum_{i=0}^{k} P_i s^i\right)^{ad} = \sum_{i=0}^{k}(-1)^i s^i (P_i)^T$$

Qualitatively, this result is intuitive: a system that is unobservable has at least one inaccessible variable and consequently cannot be controlled.

Another result is an LTV version of the elimination theorem (Ilchmann and Mehrmann, 2006, Th. 6.1), again stated below in a real-parameter version.

Theorem 9 (Elimination for LTV systems) *Assume matrix polynomials $[R(s), D(s)] \in \mathcal{A}[s]^{g\times(q+s)}$. Then there exists $R'(s) \in \mathcal{A}^{g'\times q}$ such that, for almost all $t \in \mathbb{R}$,*

$$\ker_t R' = \left\{ w \in \mathcal{C}^\infty\left(\mathbb{R}, \mathbb{R}^q\right) | \exists l \in \mathcal{C}^\infty\left(\mathbb{R}, \mathbb{R}^q\right) :\ R\left(\frac{d}{dt}\right) w = S\left(\frac{d}{dt}\right)\right\}$$

All three theorems as well as the other results presented in the article are important in that they constitute an important part of the theoretical foundation of the extension to encompass LTV systems as an approximation of nonlinear real-world systems. Due to the nature of the results as nonconstructive proofs of existence, their practical availability is limited.

3.2.3 Reception of the Behavioural Framework in Applied Mathematics

The Behavioural Framework was introduced aiming to provide a strict framework to the historically grown discipline of systems and control. This makes the BF appealing also to mathematicians working in applied mathematics, as it provides the rigour necessary in mathematics, combined with a link to applications.

A recent survey of this treatment of the BF in applied mathematics is given in (Zerz, 2008), considering itself as a 'brief guided tour' of the BF. The BF is described as appealing to applied mathematicians and theoretically inclined engineers.

The seminal work of Oberst (Oberst, 1990) serves as the basis for the development of the BF from an applied mathematics point of view, the paper itself extends the work of Willems to encompass also time trajectories that are distributions rather than functions. This extension yields the possibility to study signals that do not fulfil the requirements imposed on a function but are important also in the applications, these are e.g. impulses.

The key result of the foundational work of Willems (Willems, 1986a), the equivalence of the properties linearity, shift-invariance and completeness to the existence of a Kernel representation (Theorem 2), does not transfer readily to the continuous-time case and the problem is indeed still open. The article (Lomadze, 2007) analyses this problem.

While the initial treatment of the BF already applies algebraic concepts and structures to a large extent, the notion of a $\mathcal{D}$-module structure of the time trajectory set $\mathcal{A}$ is applied, with $\mathcal{D}$ being a commutative ring. As an example of a signal space carrying this property, the set of all smooth mappings $\mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^n)$ with the polynomial ring $\mathcal{D} = \mathbb{R}[s]$ is mentioned. The smoothness of the time trajectories is necessary as the $\mathcal{D}$-module structure requires that for any differential operator $d \in \mathcal{D}$ and $a \in \mathcal{A}$, the application results in $da \in \mathcal{A}$.

This module structure of the behaviour admits a transfer of algebraic properties to systems and control theory, resulting in two theorems and leads to a slightly different notation. The time trajectories of a single port stem from $\mathcal{A}$, the polynomial entries of the polynomial matrix from $\mathcal{D}$ and thus the behaviour can be expressed as $\mathcal{B} = \{w \in \mathcal{A}^q | Rw = 0\}$ with $R = \mathcal{D}^{g \times q}$.

The first theorem provides criteria to decide whether a system is autonomous, i.e. has no free variables.

Theorem 10 (Autonomous Behaviour, (Zerz, 2008)) *Considering a behaviour $\mathcal{B} = \{w \in \mathcal{A}^q | Rw = 0\}$ with $R = \mathcal{D}^{g \times q}$, the following are equivalent:*

1. *$\mathcal{B}$ is autonomous.*

2. *Any representation matrix of $\mathcal{B}$ has full column rank.*

3. *If $w \in \mathcal{B}$ has bounded support, then w must be identically* 0.

4. *If $w \in \mathcal{B}$ satisfies $w(t) = 0$ for all $t < 0$, then w must be identically* 0.

5. *$\mathcal{B}$ is a finite-dimensional F-vector space.*

It is claimed that a true behaviourist would choose conditions 3 and 4, as these are expressed in terms of the time trajectories.

The second theorem addresses controllability. A system is considered controllable if it admits an image representation, i.e. a representation of the form

$$\mathcal{B} = \{w \in \mathcal{A}^q | \exists \in \mathcal{A}^n : w = Ml\}$$

for some matrix $M \in \mathcal{D}^{q \times n}$.

Theorem 11 (Controllability, (Zerz, 2008)) *The following is equivalent*

1. *$\mathcal{B}$ is controllable.*

2. *Any representation matrix R of $\mathcal{B}$ is a left syzygy matrix, that is, its rows generate the left kernel*

$$\left\{z \in \mathcal{D}^{1 \times q} | zM = 0\right\}$$

of some $M \in \mathcal{D}^{q \times n}$.

3. *For any $w_1, w_2 \in \mathcal{B}$ there exists $0 < \tau \in \mathbb{T}$ and $w \in \mathcal{B}$ such that*

$$w = \begin{cases} w_1(t) & \text{if } t < 0 \\ w_2(t) & \text{if } t < \tau \end{cases}$$

4. *Any full-row-rank representation matrix $R \in \mathcal{D}^{p \times q}$ of $\mathcal{B}$ is right invertible, i.e. there exists $Y \in \mathcal{D}^{g \times q}$ with $RY = Id$.*

5. *Any full-row-rank representation matrix $R \in \mathcal{D}^{p \times q}$ of $\mathcal{B}$ satisfies*

$$\operatorname{rank}(R(\lambda)) = p \quad \forall \lambda \in \bar{\mathbb{F}}$$

where $\bar{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$.

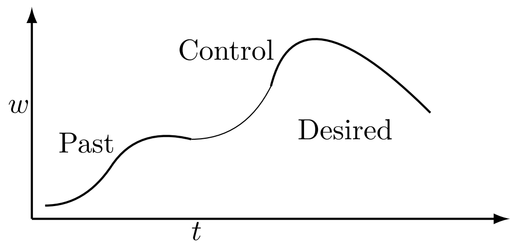


Figure 3.1: Illustration of the behavioural concept of controllability

Condition 3 represents the behavioural approach as close as possible by expressing controllability in terms of time trajectories. The condition can be interpreted as the ability to concatenate trajectories, which can be interpreted as imposing control on the system as illustrated in Figure 3.1.

This review of the BF from the perspective of applied mathematics is finalised by stating some further paths. The further paths for development are

- Multidimensional systems (cf. (Wood *et al.*, 2000, 2004))
- Continuous time-varying systems
- Discrete systems over finite rings

Both theorems transfer at least partly to multidimensional systems, the conditions 4 and 5 of Theorem 11 that do not transfer directly represent a stronger property in the multidimensional case.

Continuous time-varying systems, a topic of interest also for this thesis, can be represented in some cases by replacing the polynomials by time-varying polynomials

$$\mathcal{D} = \mathbb{R}(t)[s]$$

that may lead to signals a that are smooth with the exception of a finite set of points $E(a)$, i.e.

$$a \in \mathcal{C}^{\infty}\left(\mathbb{R} \setminus E(a), \mathbb{R}\right)$$

In this setting, $\mathcal{D}$ is commutative since for non-constant $k\mathbb{R}(t)$ the differential has to be calculated according to the product rule of differentiation.

The topic of discrete systems over finite rings is of little importance in system and control theory, but may be applied in coding theory.

3.3 Work on nonlinear systems

A significant portion of this thesis is dedicated to nonlinear systems, since nonlinearities of systems is among the key differences between real-world systems and theoretical assumptions. This topic is the subject of numerous papers, of which some will be summarised below.

Nonlinear models appear frequently in the context of chemical systems, since these are prone to exhibiting severe nonlinearities while at the same time requiring good control systems to ensure product quality. An article of a survey nature is (Pearson, 1995) that aims to structure the class of nonlinear systems. The motivation for applying nonlinear systems in this article

> '[...] comes from the unavoidable nonlinearity of the dynamics of many chemical processes. Indeed, several of the references cited in this paper deal with the nonlinearity of two of the most important chemical processes unit operations: reactions and separations.'

While nonlinear models may occur in fundamental models (e.g. a mathematical pendulum), a step frequently required for the application of nonlinear models for control purposes is to model measured data using nonlinear models. Further aspects *pro* empirical modelling are the potentially long development time, the high complexity of the resulting models and the underlying assumptions, e.g. on the importance of certain effects.

In a typical system identification procedure, an initial step is the selection of a model class from which the best model according to some optimisation criterion is selected. For nonlinear modelling, many different classes exist, of which Pearson considers Volterra models to be 'probably the best known class of nolinear systems that do possess moving average representations'. A Volterra model is represented by the difference equation

$$\begin{aligned} w_2(k) = w_2(0) + \sum_{j=0}^{\infty} a_j w_1(k-j) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{i,j} w_1(k-i) w_1(k-j) \\ + \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{l,i,j} w_1(k-i) w_1(k-l) w_1(k-j) + \cdots \end{aligned} \tag{3.4}$$

Since the sums in (3.4) are infinite and also infinite products of the variables are included in the model, to be used in applications, the Volterra model can only be applied in a truncated form.

Model classes of more practical relevance are the Wiener and Hammerstein class, presented in Section 5.2. For analytic nonlinear functions in a Hammerstein model, it is possible to represent the model in the form of (3.4) using the diagonal elements only, i.e. only those elements for which the summation variables are identical.

A generalisation of the concept of Wiener and Hammerstein models is the sandwich model, which consists of a static nonlinearity between two linear dynamic models. An even more general class is the class of block-oriented nonlinear models, that consist of series and parallel interconnections of nonlinearities and linear dynamical systems.

Another, less intuitive, model class is the class of nonlinear ARMAX (NARMAX) models if the form

$$\begin{aligned} w_2(k) =F\big(&w_2(k-1), w_2(k-2), \ldots, w_2(k-m), \\ &w_1(k), w_1(k-1), \ldots, w_1(k-n), \\ &w_3(k-1), w_3(k-2), \ldots, w_3(k-o)\big) + w_3(k) \end{aligned} \tag{3.5}$$

where F denotes a nonlinear function of $m+n+o+1$ variables and w_3 a latent variable used to express an error term. Frequently, $F \in \mathbb{R}[s]$ is assumed, yielding the advantage that in this case, (3.5) is linear in the parameters, i.e. standard estimation techniques can be applied.

A subclass of the NARMAX models is formed from the class of nonlinear additive autoregressive models with exogenous inputs which are given by

$$\begin{aligned} w_2(k) =&f_1\left((w_2(k-1)\right) + f_2\left(w_2(k-2)\right) + \ldots + f_m\left(w_2(k-m)\right) + \\ &g_0\left(w_1(k)\right) + g_1\left(w_1(k-1)\right) + \ldots + g_n\left(w_1(k-n)\right) + w_3(k) \end{aligned} \tag{3.6}$$

The f_i and g_j denote potentially nonlinear functions. A special case of (3.6) is the logistic model

$$w(k) = aw(k-1)(1-w(k-1))$$

that exhibits a chaotic behaviour. Also Hammerstein models may be expressed in terms of (3.6) by selecting $f_i(x) = a_i x$ and $g_j(x) = b_j g_0(x)$ with $g_0(x)$ being the static nonlinearity.

The introduction of the model classes mentioned above serves the purpose of addressing the model structure selection problem for nonlinear systems in a more systematic way. In general, this can be done exploratory or confirmatory. While the confirmatory approach, that selects a model structure first and the has it verified or falsified by the data, is more in line with the approach in (Willems, 1986b), a more practical approach is the exploratory. In the exploratory approach, the data is analysed first in order to find patterns hinting towards one or the other model class. The structure selection aims to find the model structure that is rich enough to exhibit the qualitative behaviour and in this way does not limit the result *a priori*. In the structure selection process according to (Pearson, 1995), a first step is the selection of the input/output structure.

The article (Pearson, 1995) yields a very systematic while not mathematically too strict approach to the problem of structure selection, that due to examples and the presentation of a multitude of model structures is appealing to the practically inclined control engineer. It does not incorporate the findings of the Behavioural Framework in that no noise disturbance on the input was assumed and an *a priori* selection of an input/output structure is required.

Another contribution by Pearson (Pearson, 2003) is a literature review aiming explicitly on the subject of structure selection. It lists the different attributes of a good model from an application view, these are considered to be:

1. Approximation accuracy
2. Physical interpretation
3. Suitability for control
4. Ease of development

While attributes 1 and 2 prefer fundamental models, attributes 3 and 4 are more distinct in low-order linear models. Consequently, in control frequently low-order linear models are applied despite their inability to model certain nonlinearities.

The subclass of nonlinear models that are useful with respect to attributes 3 and 4 have to fulfil two requirements: in order to be suitable for control, nonlinear model predictive control strategies have to exist, ease of development of a nonlinear model is strongly influenced by the availability of appropriate estimation techniques. In this respect, Hammerstein, Wiener, sandwich and bilinear models are considered appealing from the perspective of suitability for control. The system identification is relatively simple for models which are linear in the parameters, these include bilinear and polynomial NARMAX models.

Seven qualitative criteria are provided, in case at least one of them is observed on a system, a nonlinear dynamic model is required to explain the whole range of system features. These are:

- Asymmetric response to symmetric input change
- Generation of harmonics in response to a sinusoidal input without change to the periodicity
- Input multiplicity
- Output multiplicity
- Subharmonic generation, increasing the fundamental period
- Highly irregular responses to simple inputs
- Input-dependent stability

The first three features are considered mildly nonlinear behaviour, while the following three behaviours manifest a highly nonlinear behaviour. Input dependent stability is considered as a medium severe nonlinear feature.

The article is a good basis for judging nonlinearities present in a system, especially since it is written from perspective very close to application in a real-world control system. Consequently, it provides useful criteria for structure selection, allowing the user to weigh usefulness of a particular model structure against the system features required to explain with this model.

The article (Pearson and Pottmann, 2000) treats the grey-box identification of block-oriented nonlinear models. Grey-box identification is a very practical approach, as it relies on measured data and a fundamental insight into the process. In the article, it is assumed that the steady state behaviour is known to the person aiming to identify the system, a very reasonable assumption at least for the main operating points.

In (Lakshminarayanan *et al.*, 2001), bilinear models for chemical processes are identified via Canonical Variate Analysis. The author represents the view that

> the class of bilinear models represents another simple class of structures that is useful for the chemical engineer. These structures arise naturally in the chemical engineering systems. In mass or energy balance expressions, terms involving interactions between manipulated variables and state variables are usually encountered.

Two examples for the occurrence of bilinear terms in chemical processes are given:

- Feed flow rate and feed composition in a chemical reactor.
- Flow rate of fresh nutrient and the concentration of cells in a bioreactor.

This shows that the bilinear model class has significance in chemical engineering.

3.4 Simulation software

3.4.1 Introduction

In order to make a modelling and control framework appealing to engineers to apply in their domain, beside theoretical advantages, some conditions have to be met in order to make application possible. This includes a need for well usable, reliable and validated simulation tools.

The BF and its modelling methodology of tearing, linking and zooming advocate modelling and reusing of validated, modular submodels. In order to win from this methodology, a well usable and maintained simulation software with compatible version increments is required. At the same time, the ability to express behavioural equations in the most relevant of their wealth of appearances without assuming the causal direction *a priori* is a requirement for usage of a simulation tools in the context of the BF.

3.4.2 Requirements on software systems

Requirements for commercial use

The capability to develop and simulate products has significant influence on commercially exploitable features for a company. Simulation software and computer aided technologies, together with appropriate teams and processes, may reduce time-to-market, development cost and product risks. The general applicability of software tools in commercial development processes has to satisfy some needs, partly depending on the size of the enterprise and the market.

For small to medium sized companies, the user group of modelling and simulation software is typically limited to the size of the development team. The size of the company cannot sustain a software maintenance team and software tool development is not considered a core competency. In applications like this, the effort and risk in using open source or other non-established software tools is higher than the potential commercial gain. Training time has to be kept low and maintenance has to be performed by external specialists, partly even the introduction of new models is done by external service suppliers. Potentially even dominating customers require the use of certain tools.

The needs of a small to medium sized company from a software tool are typically (listed in descending priority):

- Reliable, tested and validated software
- Long-term availability of the software
- Good usability and high productivity of the user when working with the software
- Ready shipped software product with maintenance contracts
- Integration into business and development processes
- Downward compatibility of new software releases for years
- Documentation of work results
- Availability of external service suppliers on the market

Larger companies have, at least for non-specialist tools, a larger workforce using one particular product and consequently are more willing to have their own software integration and modification experts. At the same time, software companies consider large companies as very attractive customers and are offering specifically modified versions of their software, e.g. Siemens runs particular versions of SAP (Xiopia, 2012) and is a key customer for Microsoft. Typically, large companies are more able to establish their own standard and don't necessarily have to follow their customers in their selection of software tools.

The requirements on efficiency of usage and little training effort remain the same, probably not considered as important as the integration into the business processes. Thus the needs of large enterprises can be outlined as (in descending priority):

- Reliable, tested and validated software
- Long-term availability of the software
- Integration into business and development processes
- Documentation of work results
- Good usability and high productivity of the user when working with the software
- Ready shipped software product with maintenance contracts
- Downward compatibility of new software releases for years

In this way, it is more likely that a large company will use an open source software tool and has maintenance done by an internal team of experts, taking care of new releases and possibly contributing to the community. Smaller companies normally have no teams dedicated to software tools maintenance and are thus more likely to use commercial software products.

The software tools considered in this review are evaluated for the following features in order to review their commercial applicability:

INT Integration into business and development processes: interfaces to Requirements Engineering (RE) and Product Data Management (PDM) systems

USE Usability: Ready existing, relevant tools and modules, Graphical User Interface (GUI)

REL Reliability and Validation: Estimated to be roughly proportional to the number of users and duration of software availability

DOC Documentation of results: Readable, ideally graphically enhanced, execution files, database connection, version management

SUP Availability of service suppliers: *De facto* standards are expected to have higher numbers of service providers available

Requirements for use in the Behavioural Framework

Usage of a modelling and simulation tool in the BF requires the representation of models without causal directions assumed in the modelling process. An alternative to not providing this acausal modelling process is the usage of several parallel models (Ambühl *et al.*, 2010). This increases maintenance and validation effort and reduces the applicability of the efficient methodology of modelling by tearing, zooming and linking. Further required features are the reusability of submodels and the usage of implicit, equilibrium based, definitions of behavioural equations.

The evaluation criteria are thus

ACA Modelling without causal direction introduced in the modelling process

SUB Reusability of submodels

EQU Behavioural equations based on equilibria

3.4.3 Software tools

This section compares current software tools for simulation of dynamic systems from a commercial application and behavioural modelling point of view. The results are summarised in Table 3.3.

The tools available on the market can be split into three categories, one large group being based on Modelica, on group on Bond Graphs and the third category uses proprietary notations for the behavioural equations.

Proprietary tools

The market leader in the systems and control community, MATLAB (The Mathworks, Inc., 2012a) and Simulink (The Mathworks, Inc., 2012c), looks back on almost three decades of commercial development, is widely accepted and has over one million users worldwide. The significance of MATLAB goes that far that Siemens' Product Lifecycle Management (PLM) software Teamcenter offers an integration for MATLAB and Simulink (Fritzell, 2011). Also the usability and productivity of the user as well as the software reliability is considered very good.

In the MATLAB/Simulink core product, the possibility to express and run acausal models is not present, while building, maintaining and reusing submodels is well implemented. These submodels, due to their causal nature, cannot be reused in any causal context. As the Simulink package relies to a large extent on transfer functions, the behavioural equations cannot make use of equilibria.

Simulink's shortcoming of not being able to provide acausal modelling to the user is cured by a selection of add-on packages, intended for physical modelling. The basic, one-dimensional toolbox is Simscape (The Mathworks, Inc., 2012b). Since Simscape applies modelling of physical system setups, it is able to run simulations regardless of the causal direction and expresses interconnection by equilibria. The block set delivered with SimScape is extendable by specialised toolboxes (e.g. SimMechanics) or by implementation of custom elements in a high level language similar to Modelica, termed Simscape Language.

AMESim (LMS Imagine.Lab, 2012) limits its scope to one-dimensional systems specialised on simulation of pneumatic, hydraulics and control systems. This manifests itself in the present interfaces with computational fluid dynamics (CFD) software and leads to a user base mainly in the automotive and aerospace industry. Due to its capabilities to perform dynamic simulations based on finite element (e.g. CFD) models, it is possible to simulate systems before prototypes are available due to the first principles models build as accurate as possible.

No PLM integration of AMESim is known, however integration into Microsoft Office tools is provided. No trial versions are distributed without a one-day training, thus the usability can be expected to be rather poor. The results of AMESim runs can be documented well and it is even possible to give a parameterised non-editable model to e.g. sales personnel for product definition at the customer site. AMESim is well distributed in industrial and academic applications, thus a supplier base for specialist tasks can be found and also the developing company offers services for AMESim.

AMESim uses causal modelling and C code for model definition, which is only possible under the highest level software license. Due to this C code implementation, the resulting models are assignment rather than equation based. Submodels can be built from either custom made or existing blocks.

A proprietary implementation of an acausal modelling tool is 20-sim (Controllab Products B.V., 2012). 20-sim features acausal simulation of models arranged in the form of bond graphs. Custom blocks can be defined in SIDOP+, an equation definition language. The software is well usable and reliable, however business process integration via PLM software is lacking. Documentation cannot rely on additional information, grouping model and results.

Models in 20-sim can be simulated acausal and submodels can be grouped. The models are expressed in equation and equilibria are basis for interconnection of bond graph elements.

Modelica and related tools

Modelica is a declarative modelling language developed since 1997 by the non-profit Modelica Association (Modelica Association, 2012). Modelica is just a definition of the modelling language, it is not the simulation engine itself, however some requirements on the simulation engines capabilities are given.

Modelica differs from other approaches to computer based modelling in that it allows the user to define the behaviour of the system free from *a priori* concepts of the input/outout structure and in the form of equations rather than transfer functions or state space models. The equations included in the model are solved according to the current state of the simulation in terms of free variables.

As Modelica is only the language of model specification, there are a number of tools available on the market that implement simulation engines able to simulate models specified in Modelica code. The first implementation was part of the Dymola simulation software, which is now part of the CATIA environment. Other commercial implementations include Wolfram System-

Modeler, SimulationX and MapleSim. There are also free implementations of Modelica, with the most notable being jmodelica.org, OpenModelica and Scicos.

Dymola (Dassault Systèmes, 2012) is a graphical implementation of the modelica language and has recently been integrated into the Dassault Systèmes software suite, that is based on the computer aided design (CAD) software CATIA, but also includes the PLM software Smarteam. For this reason, the integration into development process, documentability as well as reliability are to be considered as very good. Usability of Dymola can be expected to be good, although the wealth of features (e.g. integration with 3D CAD) will require some training.

The Dassault Systèmes software bundle can be considered a *de facto* standard in the automotive industry, so that exchange of data within the supply chain is possible, also a number of service providers for modelling, validation etc. offer their services on the market.

Due to its implementation of the Modelica language, Dymola is capable of acausal simulation, the block-oriented formation of submodels and is an equilibria based tool.

While Dymola, due to its good integration with 3D CAD systems, aims on mechanical engineers as users, Wolfram SystemModeler (Wolfram Research, 2012) with its integration into the Mathematica software is more targeted on system engineers having a closer link to applied mathematics. The integration with Mathematica enables a good documentation in the form of a mixture of Mathematica/Modelica code, results and formatted text. An integration into PLM or PDM software is not provided, however the Mathematica tools are well usable and reliable.

Wolfram SystemModeler and its predecessor, MathModelica, are not very popular, thus only a limited number of consultants specialising in Wolfram SystemModeler are to be expected. Due to the Modelica implementation it may be possible to find support among the Modelica modelling service providers.

Thanks to the usage of Modelica and the related interpreter, Wolfram SystemModeler is a tool that provide graphical arrangement of blocks for simulation. The blocks can be defined either causal, i.e. in the form of assignments, or acausal, i.e. in the form of equations and it is possible to form subsystems.

A simulation tool without company affiliation to either CAD or mathematical software that can be considered mainstream is SimulationX (ITI GmbH, 2012). The software is not integrated into a PLM system but features numerous external interfaces to industry specific software, e.g. CAD, FEA, CFD. This reduces the effect of being locked into any vendor specific software, but is prone to make handling of these interfaces more difficult.

SimulationX focuses more on the functionality with its plenty of interfaces and vast libraries than on documentation of the results, this has to be done in external software. ITI GmbH and its subsidiaries worldwide offer support with the software, further Modelica-related services are to be found more often. The software is compatible with Modelica and features acausal, equation-based simulation and also the generation of submodels in the object-oriented paradigm.

Integrated into the Computer Aided Algebra software Maple is MapleSim (Maplesoft, 2012) that combines the computational power and usability of Maple with acausal, equation based modelling features of Modelica. The same as for MathModelica, the documentation within the Maple worksheets is excellent but the format is not maintained in any of the current PLM software.

Reliability of this mature software package can be expected to be very good. MapleSim has a growing user base mainly in the automation and aerospace industry and Maplesoft offers modelling and other MapleSim related services.

The most important free software packages implementing Modelica are JModelica.org (JModelica.org, 2012), OpenModelica (OpenModelica, 2012) and Scicos (INRIA, 2012). These tools are all fully operational, but each one has a particular drawback that puts the commercially available tools in a favourable condition unless the user can accept that drawback. For all three, no integration into PLM/PDM software is known.

JModelica.org is a free implementation of the Modelica engine without graphical user interface, instead all user interaction is executed via Python, a scripting language. This leads to reduced integration into development processes, difficult documentation and bad usability. Further it cannot be expected that many companies offer support for JModelica.org. The software is developed for extended use via scripting in applications as optimisation or model calibration. For these purposes, a high reliability is valued higher than good usability.

Scicos is part of the Scilab package developed by INRIA. It implements some Modelica blocks, however these are still handled in a causal manner. Scicos can use Scilab for scripting and documentation and due to its relatively frequent usage in the scientific community, it can be expected to work reliable. Support appears to be hardly available. The Modelica blocks can be grouped to submodels and are equation based.

OpenModelica features a textbook based implementation of the Modelica language which can group code, results and text for easy documentation. The models can only be defined and arranged in the form or Modelica code, which is difficult and error-prone. The software is relatively reliable, however a large user community for testing and validating is not obvious. This also may lead to little external support for users, although the usage of the open Modelica standard in its purest form improves this situation. The software, due to its full Modelica implementation, features acausal, equation based modelling and object-oriented grouping of models.

3.4.4 Summary

The single software tools for simulation in the BF exhibit their feature as outlined in Table 3.3. It is not possible to select the best tool, as depending on industry, application, user group and budget, trade-offs have to be made.

3.5 Interdisciplinary topics

Since modelling and simulation as well as causality do not only play a role in control engineering; and also the value of mathematical frameworks is estimated in other scientific disciplines, this section presents some relatively close interdisciplinary view on the subject.

An interesting light on 'the unplanned impact of mathematics' is shed in (Rowlett, 2011). This unplanned impact is frequently caused by the intervention of tools before their purpose is known or can be envisioned. From the view of real-world applications, this often becomes obvious by the age of the mathematical tools. It is not rarely the case that currently used tools were initially developed decades or centuries ago. Part of the reason for this and further advantage of the vast set of mathematical tools is that once

	INT	USE	REL	DOC	SUP	ACA	SUB	EQU
MATLAB Simulink	+	+	+	+	+	−	+	−
Simscape	+	+	+	+	−	+	+	+
Dymola	+	+	+	+	+	+	+	+
AMESim	−	−	+	+	+	−	+	+
20-sim	−	+	+	−	−	+	+	+
Wolfram SystemModeler	−	+	+	+	−	+	+	+
SimulationX	−	−	+	−	+	+	+	+
MapleSim	−	+	+	+	+	+	+	+
JModelica.org	−	−	+	−	−	+	+	+
Scicos	−	+	+	+	−	−	+	+
OpenModelica	−	−	+	+	−	+	+	+

Table 3.3: Comparison of software tools for modelling and simulation. + denotes well developed features, – denotes absent or little usable features.

their correctness is proven, there is no need for reevaluation and, moreover, no chance of falsification:

> 'If it was true for Archimedes, it is true today.'

However, as a well-known fact, mathematics is not only driven by application and while many mathematicians do value theoretical work higher than applicable, this (currently) inapplicable part of mathematics contributes to the beauty and magnificence of mathematics.

Reflected on the topic of this thesis, this may be interpreted such that the applicability of the BF to real-world problems is indeed desirable, but if the BF proves of little use for today's applications, this does not imply that it will be of limited use forever.

As mentioned in Chapter 2, the issue of causality in modelling is not only of interest in the systems and control community, even the 2011 Nobel price was awarded for work analysing the causality from observed data instead of assuming it *a priori*. The close relationship between econometrics and systems and control leads to the application and extension of advanced modelling and control topics, as e.g. robust control in (Hansen and Sargent, 2001).

One of the 2011 Nobel laureates, Christopher A. Sims, began to put into question the process of assuming the causal relationship of two variables before starting the identification process in early publications (Sims, 1972, 1980). His co-laureate, Thomas J. Sargent, works into the same direction, also for many decades, as in (Sargent and Wallace, 1975; Sargent, 1977).

The doubt cast on causality assumed *a priori* is partly due to the type of experiment, which in the socio-economic research mostly cannot be an experiment specifically conducted for the purpose (imagine a PRBS test on money supply). Instead, typically accidental experiments are used for calibration of econometric models. Since in these experiments, a causal direction is even less obvious to assume than in technical experiments where the system is mostly intended to perform some input-output operation and the experiment is conducted in some designed fashion, tests for causality and a clear separation between correlation and causality emerged comparably early and were more readily adopted.

In (Kim *et al.*, 2008) (coauthored by Sims), a strong case is made for nonlinear models, claiming that linear models are not always enough, as they mainly aim for local approximation. As a starting point for this paper, variables, as in the BF, are not separated into input and output. As an argument for this reduced set of assumptions, the size of systems and models under consideration in the econometrics is provided.

Another parallel to the BF is the formulation of kernel-like, nonlinear model structures. To these nonlinear model structures, a Taylor series expansion is applied prior to parameter estimation.

The assumptions on the level of *a priori* knowledge to be applied in modelling as well as the resulting model structures resemble partly the structures introduced in Chapter 5.

An interesting view on the effects of wrong socio-economic models is reported in (Cogley and Sargent, 2005), posing the question whether the comparably bad inflation-unemployment outcomes in the US in the 1970s were bad luck or bad monetary policy.

An assumption of previous work on this question (DeLong, 1997) is that the economic models of that time showed an exploitable trade-off between inflation and unemployment. Later, this model was revised and inflation-unemployment outcomes were improved.

While in the paper, no causality is assumed *a priori*, stability is assumed for the developed models, based on the view that the Federal reserve bank (Fed) chooses its policy in a purposeful, i.e. stabilising, way. The causality

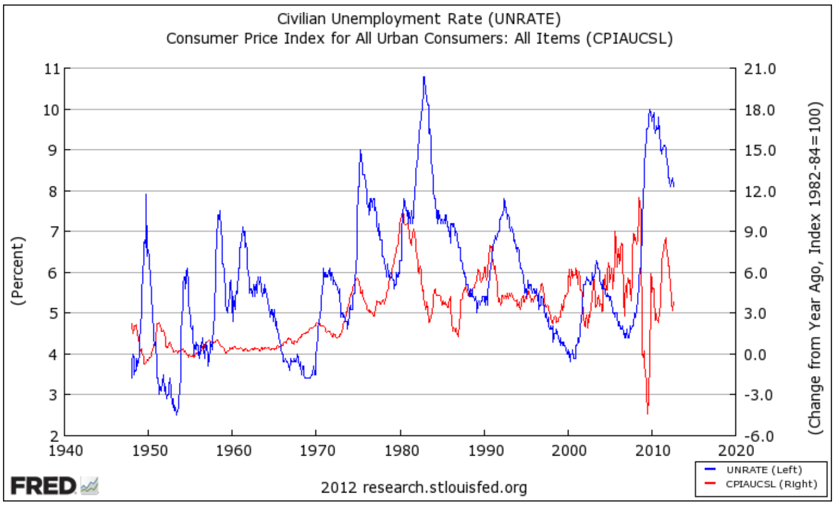


Figure 3.2: Unemployment (left) vs. Annual Inflation graphs (right). Source: research.stlouisfed.org

assumption is dropped in this work since Fed cannot base its current quarter decision on the (then unknown) unemployment rate of that quarter. With these assumptions, one of the conclusions is that possibly due to an erroneous model the responsible chairman of that era produced the greatest peacetime inflation in the US history - with other possibilities for his decisions being insuffiecient patience and inability to commit.

This work shows how much can be lost by assuming the wrong causal dependence and basing control on this wrong assumption. In the case at hand, the variable available for control is the inflation (via the money supply), the to-be-controlled variable is the civilian unemployment rate. The model leads to the assumption that it is possible to maintain the desired low unemployment rate with low inflation, achieved by this control was in fact a steady rise in unemployment. In the years after the model change, the duration of increasing unemployment rates was effectively limited by increasing money supply, at the cost of higher inflation, as shown in Figure 3.2.

3.6 Concluding remarks

The present chapter presents a survey of literature and software related to the aim of this work. This survey starts with the foundational works, reviewing model structures and the existence theorem for an (AR) representation. The further steps follow the logic of Willems tripartite paper (Willems, 1986a,b, 1987), which establishes exact modelling before addressing the approximate modelling case.

Further developments of the BF relevant for this thesis include the very practically oriented modelling methodology of modelling by tearing, zooming and linking. Also works on control and adaptive control are reviewed with an application to real-world systems in mind. In order to establish the view from both sides, also the reception of the BF in applied mathematics is surveyed.

Since the real-world systems this thesis aims to apply behavioural techniques to are partly distinguished by their nonlinearity, work on nonlinear systems is reviewed. Especially two works that analyse subclasses of the vast class of nonlinear systems, review their qualitative behaviour and show methods for selection of the appropriate nonlinear model structure are presented. The qualitative properties and the selection of the appropriate model structure is a topic that is of interest in the following chapters.

The systems and control related part of the survey of related work is finalised by the review of software tools. For this purpose, requirements arising from the application in industry as well as from the acceptance of the BF as modelling framework are collected and the software tools are measured according to these criteria. This enables the selection of the optimal software for application depending on industry, users and budget.

Some interdisciplinary topics dealing with the impact of mathematics and with causality in the social sciences put this thesis into a more integral setting.

Chapter 4

Modelling from physical principles

> We have no right to assume that any physical laws exist, or if they have existed up to now, that they will continue to exist in a similar manner in the future.
>
> Max Planck

4.1 Introduction

This section analyses the development and range of applicability of models that are derived from physical laws or comparable laws in the respective domain. Models derived from such laws are frequently encountered in technical applications due to their advantages of being an understandable approximation of the system to be represented and derivable for engineers not or little skilled in the mathematical tools for system identification.

From these basic properties stem some advantages further along the system engineering process. Models from physical principles can be consistently derived from the system properties, although they are not unique due to unavoidable simplifications in the modelling process. This derivation can then be followed easily by engineers at customers or notified bodies when it comes to customer acceptance or homologation of the system under consideration based on the data gained by simulation.

While philosophically only the empirical model derived from observations exists, practically a very common approach is to break the system down into a set of interconnected subsystems for which a model can be found (Albertos and Mareels, 2010). These subsystem models effectively constitute empirical models, as also the laws today considered as basic laws are derived from observation. As prominent examples, consider the laws of Newton and Hooke in the mechanical domain or Ohm's law in the electrical. These were derived from observation of the object under study.

The application of fundamental laws forms a system oriented approach in that the subsystems are not modelled new each time, but are in a sense validated over centuries of application without falsification. These validated subsystem models are not able to explain unexpected behaviour, however the analysis represented in a model helps to analyse reasons for such effects. At the same time, the use of first principles models may reveal behaviour at operational extremes, such as crash scenarios, that cannot be tested in some industries such as railways due to prohibitive cost.

The development of models from physical principles often makes use of graphical representation forms in order to systematically derive the subsystem models and collect the information on their interconnection. In this chapter, the common applicability of bond graphs and behavioural modelling leading to a simplified deduction of system models will be presented.

While this is not the only appropriate graphical model representation, the acausal nature and the power conserving property makes bond graphs very appealing in the context of the BF.

4.2 Abstraction and interpretation

4.2.1 Abstraction

Real world systems of different complexity are modelled in many ways, regardless whether this process of modelling is mentioned explicitly. The resulting models may range from mental models of users interacting with e.g. a central heating to elaborate mathematical models of industrial plants used for performance optimisation. Common to all these models is the abstraction of real world systems into some formulation explaining the aspects vital for the intended purpose.

In (Albertos and Mareels, 2010), a model is considered a partial representation of a system's dynamic behaviour. The system cannot be mapped uniquely onto a model, additional coefficients such as the level of approximation play a role in model selection.

From a mathematical viewpoint, the process of abstraction limits variables and parameters in number and range as well as function spaces of possible time trajectories, to mention but a few. Usually the number of system variables will be limited in order to keep track and enable system identification, the system is assumed to be a lumped parameter linear system, limiting possible solutions to some function spaces. The parameters will be assumed to be constant over time and for numerical reasons within some range.

Some of these decisions are made based on the intended purpose, thus interacting with the interpretability of the model. Further decisions are made as no tools are available for the necessary tasks, others are made as no experience is present with different modelling techniques.

In the case of regenerative systems, systems with power flow inversions, frequently control engineers apply their well known methods, which distinguishes between input and output, resulting in a set of partly valid models which are combined to cover the range necessary. However, these models are not proven to behave in the desired way, in fact, they might even exhibit a behaviour contrary to the modelled system.

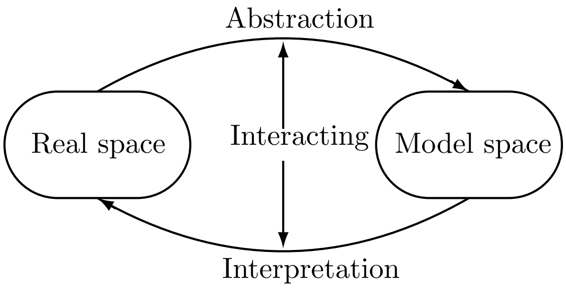


Figure 4.1: Interpretation and abstraction

4.2.2 Interpretation

Once a model is found, the results are projected back onto the real world system. Sometimes the model turns out to be not applicable for the intended purpose, which may be due to limitations made earlier in the abstraction process, incorrect or incomplete data gathered for modelling or the system not exhibiting the desired behaviour, among others.

The modelling task may thus be considered as a process of abstraction followed by interpretation, where both parts of the process may be adjusted in repeated cycles. As usually the purpose of modelling defines the necessary interpretation, mostly the abstraction is adjusted. This process and the interaction is depicted in Figure 4.1.

4.2.3 Validity of models

With respect to modelling for purpose, the validity of models can be defined as the interpretability for the intended purpose. If, for example, stability analysis is the purpose of a modelling task, a multiple model structure is in general not interpretable for this purpose. Similarly, when the model is derived with the use of *a priori* assumptions on e.g. linearity, time invariance or causality, these assumptions may restrict the validity of the model in that it is prone to show a behaviour within the assumed restrictions.

One method frequently applied in order to extend the validity of models is the extension of the model class to encompass nonlinear models, as proposed in e.g. (Pearson, 2003). Also time varying linear or nonlinear models are rather accepted in science and technology. The application of the behavioural modelling methodology to systems that may exhibit inversions of causality is lacking far beyond that of nonlinear methods.

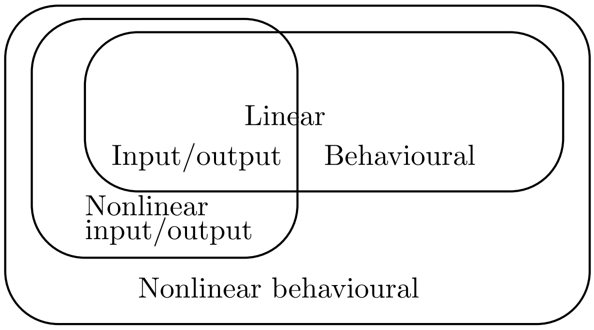


Figure 4.2: Venn diagram of system models

The validity of models may be interpreted in a set theoretic manner: the superset of models consists of the class of nonlinear behavioural models, i.e. nonlinear models without assumed causality. This superclass contains nonlinear input/output models as well as linear behavioural models, these subclasses impose a limitation of validity in terms of linearity or causality assumed *a priori*. The class 0f linear behavioural models contains the class of linear input/output models. The Venn diagram in Figure 4.2 indicates that with a small subset of models that is applied and taught, we seek to explain a wealth of phenomena present in real world systems.

In the remainder of this chapter, methods for extending model validity by reducing *a priori* assumptions will be investigated and developed. As a first stage, the practical relevance of dropping the causality assumption on linear systems will be shown by examples. At the same time, graphical modelling by help of bond graphs will be reviewed from the perspective of application to real world systems.

4.2.4 Graphical model representations

The modelling task can be performed in a multitude of representations. Essential to the modelling process for engineering is the usage of an appropriate graphical representation of the model.

Graphical representations are popular among practising control engineers, some of the reasons for this acceptance is the facilitation of communication as well as the potential similarity to the system to be modelled. Further, it provides a means of documentation and verification, both of great importance in an engineering world increasingly bound by regulations. The

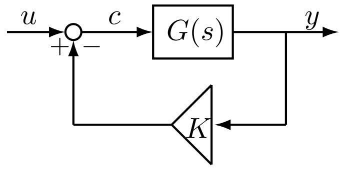


Figure 4.3: Block Diagram Example

appropriate grapical representation can thus be considered crucial for the acceptance of a theoretical framework.

As discussed in Section C, the block diagram, which separates signals into input and output, dominates both modelling and simulation. In (Willems, 2000), graphical representations in terms of undirected graphs are considered advantageously, whereas in general, bond graphs (Karnopp and Rosenberg, 1968; Karnopp *et al.*, 1990) are more widely used in modelling and simulation of physical systems.

Another approach for system modelling, partly driven by the Modelica development, is the representation in the form of abstracted physical assemblies. This typically includes all major elements of dynamical systems in a particular domain, e.g. resistors, capacitors and inductances in the electrical domain, as well as some nonlinear elements such as diodes. In this sense, the approach to model a system in its physical abtraction is similar to the bond graph methods, however less abstract.

Block Diagrams

The omnipresence of the block diagram, requiring *a priori* assumptions of the signal flow direction, in teaching and applications reflects the need for causality popular in humans as highlighted in Section 2.3.

Block diagrams consist of one or more labelled rectangular blocks, representing the subsystems (or possibly the overall system), and directed lines. Beside this, it is popular to deviate from the rectangular shape for the summation (adding its inputs to form the output) and the amplifier (multiplying its input by a fixed factor), to a circle and a triangle, respectively.

In Figure 4.3 a basic example shows the important elements: $G(s)$ denotes an arbitrary subsystem, K a multiplicator element and the circle between u and c the summation element. The signals u and y do not terminate in a block and are considered external signals, while c is an internal signal.

Also in Figure 4.3, the directional flow of signals becomes obvious. The signals that are exchanged between the blocks are following the direction of the arrows; the signals are processed in the blocks according to transfer functions. These drawbacks let block diagrams appear less suitable for application in the behavioural framework, despite its popularity and availability of a wealth of tools and techniques.

Bond Graphs

Taking into account the aspects mentioned in Example 2, the acausal structure of bond graphs may be considered advantageous for a graphical representation in the behavioural framework. The notion of bond graphs and the possible bidirectional flow of power along the bonds fits well into the behavioural framework as well as the fact that on each bond two variables are present, effort and flow. This presence of two variables on each bond fits well into the view of modelling by tearing, zooming and linking (Willems, 2008a). A further similarity to the behavioural framework is that the representation is based on equilibria, such as Newton's or Kirchhoff's laws.

Bond graphs were established in (Paynter, 1961), further development took place in (Karnopp and Rosenberg, 1968) and (Karnopp *et al.*, 1990). An object oriented view on modelling in terms of bond graphs is presented in (Damic and Montgomery, 2003) while a recent introductory article can be found in (Gawthrop and Bevan, 2007).

The elements of bond graphs are bonds, two kinds of junctions, five different components and two sources, yielding a comprehensive set for modelling of linear systems, which may be extended to cover nonlinearities.

The bonds connect the components and transfer power, which is the product of two variables, generally termed effort e and flow f. This pair of variables relates to the physical world depending on the domain, for mechanical translational movements, effort corresponds to force while flow represents the velocity. In addition to these direct variables, internal variables acting like states are generated by accumulation. These are called momentum and displacement and are accumulated by integrating effort and flow over time, respectively. In some domains, these have a physical meaning, while in oth-

Domain	Effort	Flow	Momentum	Displacement
Mechanical Translational	Force	Velocity	Momentum	Displacement
Mechanical Rotational	Torque	Velocity Angular	Momentum Angular	Angle
Electrical	Voltage	Current	Flux linkage	Charge
Hydraulic	Pressure	Volume Flow	Pressure Momentum	Volume
Thermal	Temperature	Heat flow	-	Heat energy

Table 4.1: Bond graph variables in various domains

Component	Equations
Inertia I	$e = I\dot{f}$
Capacitor C	$f = C\dot{e}$
Resistor R	$e = Rf$
Transformer TF	$e_0 = ke_1,\ f_1 = kf_0$
Gyrator GY	$f_0 = ke_1,\ f_1 = ke_0$
1-junction 1	$\sum_{i=1}^{n} e_i = 0,\ f_1 = f_2 = ... = f_n$
0-junction 0	$\sum_{i=1}^{n} f_i = 0,\ e_1 = e_2 = ... = e_n$

Table 4.2: Components of bond graphs and governing equations

ers they are abstract values. These variables for popular technical domains are given in Table 4.1.

The topology of bond graphs is determined by equilibrium conditions rather than input/output assignment. In order to be able to solve the resulting equations, the sign of the variables on that bond has to be known. For this purpose, a half arrow on one end of a junction represents this sign convention. In the direction of the half arrows the power is considered positive.

The components comprise of a resistor, an inertia and a capacitor as basic components. These basic components have one port to be connected to and relate the variables according to their governing equations. The energy storing components, capacitor and inertia, require a state to denote the amount of energy present in the system. These states are termed generalised momentum for the inertia and generalised displacement for the capacitor. The three basic components and their respective equations are given in Table 4.2.

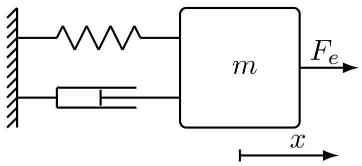


Figure 4.4: Mass-spring-damper system

The Transformer and the Gyrator offer two ports and perform a power conserving transformation of effort and flow. In Table 4.2, the variables of port i are denoted e_i and f_i for effort and flow and k for the transformation constant.

To connect the components, only two kinds of junctions are necessary, the 0-junction and the 1-junction. In a 0-junction, efforts are equal and flows sum to 0, which is equivalent to e.g. Kirchhoff's current law, whereas in a 1-junction, efforts sum to 0 and the flows are equal, which relates to Kirchoff's voltage law or Newton's second law, for example.

In addition to the above components, sources are introduced as a source of effort and a source of flow. The source of effort relates to a force in the mechanical translational domain, the source o flow consequently a displacement at a certain velocity.

It is common to denote the parameters of the components in bond graphs following the component type, divided by a colon, e.g. I:m for an inertia representing a mass m.

Example 5 *A mass-spring-damper system as shown in Figure 4.4 is represented by the corresponding bond graph.*

The mass-spring-damper system contains a mass, represented by an inertia I, a spring, represented by a capacitor C and a damper, represented by a resistor R. It interfaces with an external force, represented by a source of effort SE:F_e, which may drive the system or which may be caused by the system. Two further forces act on the mass, caused by the spring of stiffness c, $F_s = -cx$ and by the damper with friction coefficient b, $F_d = -b\dot{x}$.

By Newton's second law, the sum of all forces in x-direction is propor-

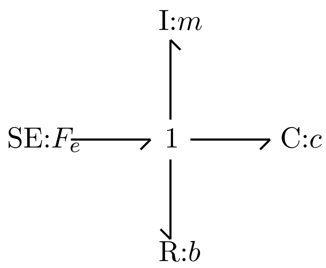


Figure 4.5: Mass-spring-damper system as a bond graph

tional to the acceleration is this direction, yielding

$$m\ddot{x} = \sum F = F_e + F_s + F_d = F_e - cx - b\dot{x} \tag{4.1}$$

Connecting the components to a 1*-junction completes the bond graph, which is a representation of* (4.1) *without any causality implied. The resulting bond graph is shown in Figure 4.5.*

Modelling in physical abstractions

The simulation of models represented in the form of physical abstractions of a system is possible with the help of software packages discussed in Chapter 3. These packages such as MapleSim or SimScape are frequently based on the Modelica language or derivatives, offering a model based on equilibria instead of signal conversions. These equilibria are formulated in two variables depending on the domain, similar to the concept of effort and flow for bond graphs.

The major advantage of this approach is that the model is not reduced to the very abstract level of only 5 components, not directly linked to their physical representations, and the topology of the system can be gained by connecting elements without the derivation based on only two concepts of junctions.

A drawback of physical modelling is that the system assembled in this way may contain equations that are hard or impossible to solve numerically and that transformations between the individual domains have to be performed explicitly.

A screenshot of an electrical network consisting of a resistor, a capacitor

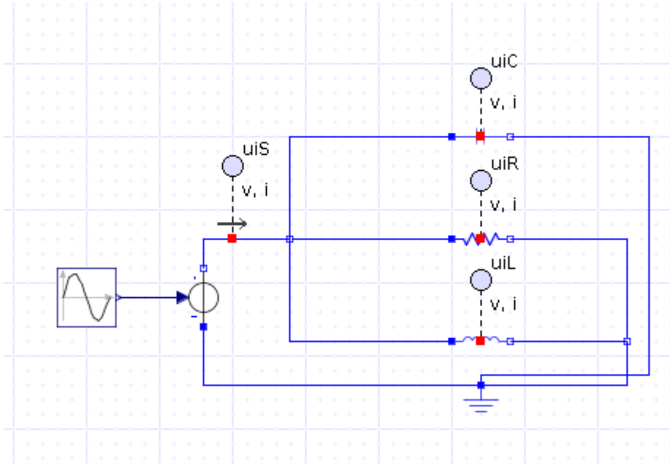


Figure 4.6: RLC system represented as a physical abstraction

and an inductance (RLC system) represented in MapleSim is given in Figure 4.6

4.3 Acausal linear first principles modelling and simulation

4.3.1 Introduction

The most common modelling approach in application and education is that of linear time invariant models with an *a priori* defined input/output structure. This is typically done by utilising block diagrams and transfer function models and is supported by numerous software tools. A different approach is to represent the system graphically as a bond graph or in its abstract physical representation. This approach is acausal and offers all advantages of modelling from physical principles, namely to yield models in early design stages that are easy to understand even for engineers without particular skills for empirical modelling.

While modelling with a linearity assumption may be sufficient for many applications, especially for first studies on the feasibility of concepts or performance estimates, frequently the causality assumptions limits the interpretability of models to a degree that calls for a different approach. An example for such a system is given in Section 4.3.2, the model of a train

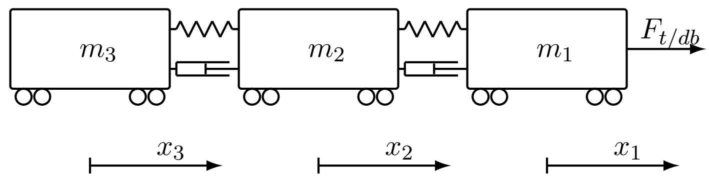


Figure 4.7: Three vehicle train consist

longitudinal dynamics under tractive effort. This dynamical behaviour is important since it indicates the loads introduced into the carbody, the derailment stability and the passenger comfort. All three properties are high level requirements and should be verified in the earliest possible design stage.

4.3.2 Application to longitudinal train dynamics

Assume a coupled train consisting of three vehicles, one of these (m_1) being the locomotive while the remainder are coaches without traction effort available, as depicted in Figure 4.7. This rake of three vehicles is representative for all possible train consists, which can be built by increasing the number of center cars m_2. Between the vehicles, a simplified model of an automatic coupler is assumed, consisting of a spring with coefficient c_i and a damper with damping b_i between vehicles i and $i+1$. This model does not consider any motion orthogonal to the longitudinal movement of the train and assumes a stiff, lumped mass car body, which is a common simplification (Cole, 2006).

A force $F_{t/db}$ for traction or dynamic brake efforts is assumed to be acting on the leading vehicle m_1, no grade, retardation or pneumatic brake forces are considered to be acting on the coaches. The longitudinal movement and dynamics can be expressed as

$$m_1\ddot{x}_1 + b_1(\dot{x}_1 - \dot{x}_2) + c_1(x_1 - x_2) = F_{t/db} \quad (4.2)$$

$$m_2\ddot{x}_2 + b_1(\dot{x}_2 - \dot{x}_1) + b_2(\dot{x}_2 - \dot{x}_3) + c_1(x_2 - x_1) + c_2(x_2 - x_3) = 0 \quad (4.3)$$

$$m_3\ddot{x}_3 + b_2(\dot{x}_3 - \dot{x}_2) + c_2(x_3 - x_2) = 0 \quad (4.4)$$

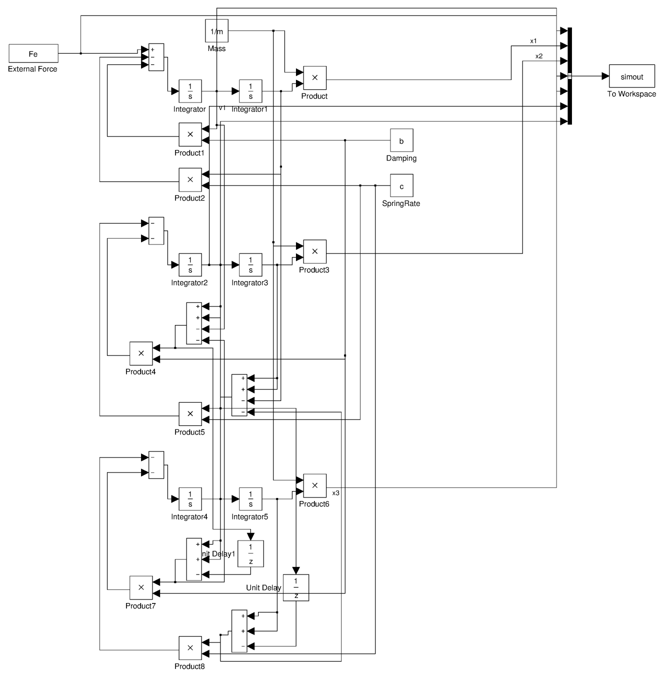


Figure 4.8: Three vehicle train consist in block diagram representation

For a further simplication, assume $m_1 = m_2 = m_3 = m$, $b_1 = b_2 = b$ and $c_1 = c_2 = c$, yielding

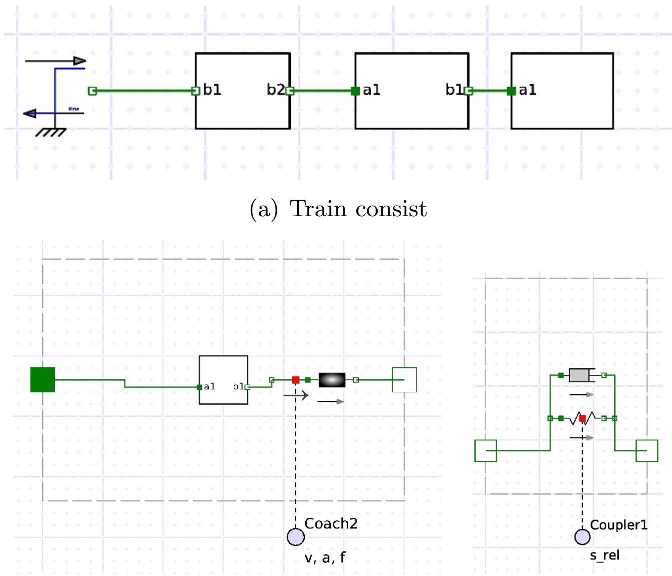
$$m\ddot{x}_1 + b\,(\dot{x}_1 - \dot{x}_2) + c\,(x_1 - x_2) = F_{t/db} \tag{4.5}$$

$$m\ddot{x}_2 + b\,(2\dot{x}_2 - \dot{x}_1 - \dot{x}_3) + c\,(2x_2 - x_1 - x_3) = 0 \tag{4.6}$$

$$m\ddot{x}_3 + b\,(\dot{x}_3 - \dot{x}_2) + c\,(x_3 - x_2) = 0 \tag{4.7}$$

Equations (4.5), (4.6) and (4.7) form a system of differential algebraic equations (DAE) that can be expressed as a block diagram. The resulting block diagram, shown in Figure 4.8, exhibits the complex differential algebraic structure.

The same system can be expressed as its physical abstraction in an appearance very similar to Figure 4.7, depicted in Figure 4.9 as screenshots from MapleSim. The difference to the sketch is that the physical abstraction can be grouped hierarchically into submodels, improving the reusability of the submodels and that probes have to be attached to measure variables of interest.



(a) Train consist

(b) Coach submodel

(c) Coupler submodel

Figure 4.9: Three vehicle train consist (a) with coach (b) and coupler (c) submodels in physical abstraction representation

The simulation in MapleSim generates the overall system equations based on the component equations formulated in Modelica and solves them based on the excitation, in this case the tractive effort F_t. In this approach, the DAEs can be resolved and the simulation admits acausal flow.

In Figure 4.10 the resulting acceleration of a simulation for $m = 90000$kg, $c = 500\frac{\text{kN}}{\text{m}}$ and $b = 1\frac{\text{kNs}}{\text{m}}$ in MapleSim is shown. This simulation shows a clearly undesirable behaviour by yielding a large backlash, that is even strong enough to decelerate the leading vehicle despite the high tractive effort of $F_{t/db} = 100$kN.

This backlash leads to a force higher than the tractive effort available of the locomotive. At first sight, it may seem acceptable to design the strength of the coupler system to bear the full traction force of the locomotive, however the simulation of the train dynamics exhibits forces as in Figure 4.10 which are 20% than the available tractive effort. Thus a coupler would be overloaded and may at least exceed its fatigue strength. Further such oscillating acceleration at the application of tractive effort will lead to a reduced ride comfort for the passengers.

In addition to the overload of the coupler and the reduced ride comfort,

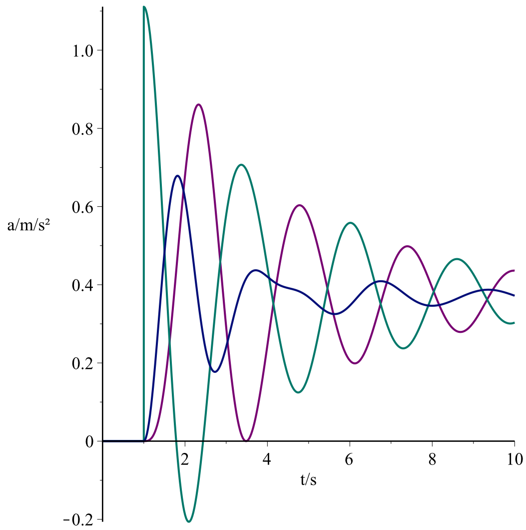


Figure 4.10: Acceleration result of an acausal simulation of the three car train set with low damping, leading vehicle in green, m_2 in purple and m_3 in blue

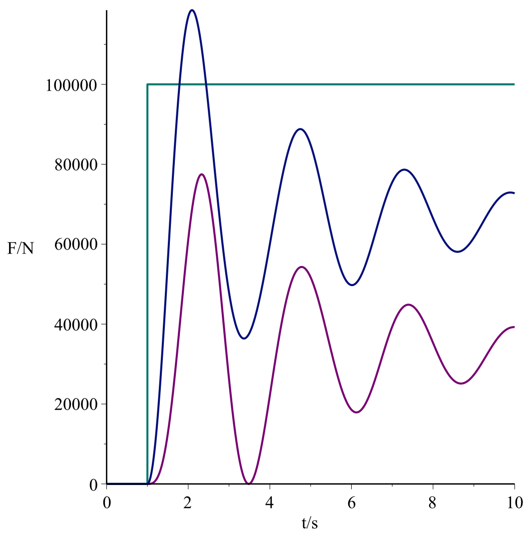


Figure 4.11: Force result of an acausal simulation of the three car train set with low damping, leading vehicle in green, m_2 in blue and m_3 in purple

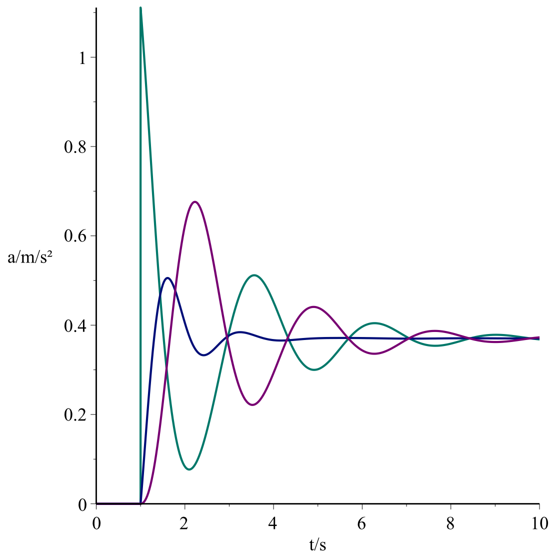


Figure 4.12: Acceleration result of an acausal simulation of the three car train set with higher damping, leading vehicle in green, m_2 in purple and m_3 in blue

the increased force may lead to wheel slip of the locomotive, since for a 90 ton vehicle a force of 120 kN yields a utilisation of friction coefficient of $\mu > 0.13$. Under certain circumstances such as autumn rail conditions, this may be unavailable and consequently the wheel would slip. Thus an increase in coupler strength is not a viable option.

By inspection of the overshooting behaviour of Figure 4.10, an alternative is the selection of a higher damping coefficient. If the damping coefficient is selected as $b = 1000\frac{\text{kNs}}{\text{m}}$, the result does not exhibit any further deceleration despite some further undulating behaviour. Also the forces are lower than before and do not exceed the tractive force of the locomotive, solving the problems of yield strength of the coupler and the wheel slip in a common approach. The simulation results are shown in Figures 4.12 and 4.13

The detection of this force overshoot in early design stages, possibly already during conceptual design stages, offers possibilities to change the behaviour at very low cost, reduced extra design effort and risks to project schedules. For this reason, it is interesting to note that the block diagram in Figure 4.8 cannot be simulated in Simulink since the algebraic loops cannot

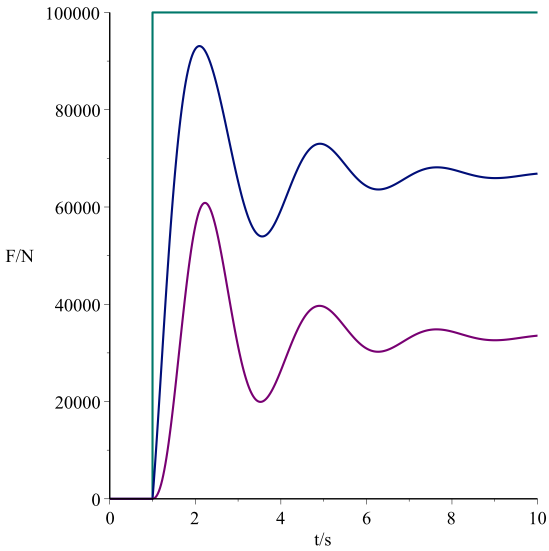


Figure 4.13: Force result of an acausal simulation of the three car train set with higher damping, leading vehicle in green, m_2 in blue and m_3 in purple

be resolved. Simulink can cope easier with algebraic loops when dynamic Simulink blocks are contained in the loops. In order to obtain a simulation at all, delays of 1 ms were introduced in the loops, yielding an executable Simulink model.

The model however shows far different results, the undulating force and acceleration behaviour is almost invisible. Coach m_3 appears not to be accelerated at all and especially the coupler force higher than a potential design force is not shown in this simulation. The result is given in Figure 4.14.

This example highlights the facts that causal block diagram representations of a system tend to be more complex in the case of multidirectional interconnection and that models have to admit by design the statements they are used for. In the present case, a designer could have trusted the results of the causal simulation and may have designed a vehicle coupler that does not fulfil its specification.

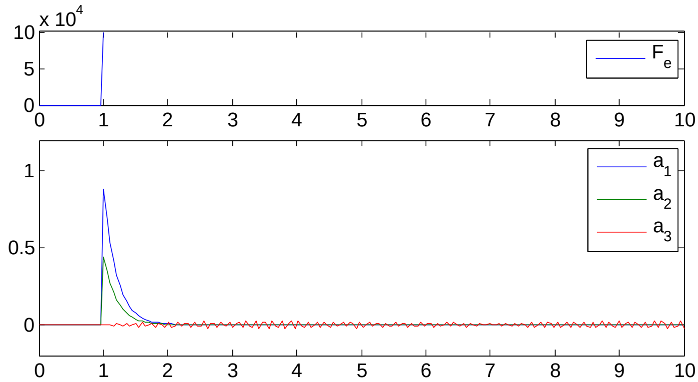


Figure 4.14: Acceleration result of a causal simulation of the three car train set with low damping, leading vehicle in blue, m_2 in green and m_3 in red

4.3.3 Application to control

While the representation of a system as its physical abstraction is very appropriate for simulation environments using the Modelica language, bond graph representations may be useful for analytical calculations. These analytical calculations have certain advantages over simulations, for example they can be generalised to yield quick estimates of performance e.g. in the bid phase of a project.

This section provides an example of a process for deriving the manifest behaviour of an interconnected system using bond graphs.

The system under consideration as the plant is a mass-spring system as conceptually depicted in Figure 4.15(a). This system has variables in the mechanical translational domain and from Table 4.2, it is possible to find the relationship between elements of mechanical translational systems and the respective bond graph. In the bond graph representation of mechanical translational system, the concept of effort is used to explain forces while the concept of flow relates to the velocity.

The systems consists of one spring and a mass. The mass element is expressed by help of the generalised inertia I while the spring element is replaced by the generalised capacitor C.

Further two sources of effort SE, namely the forces F_e and F_c are interconnected. For physical reasons, the velocities of the interconnected elements have to be equal on the point of their interconnection and the forces sum up to 0. This is represented by the 1-junction in the bond graph in Figure 4.15(b).

The graphical representation can be transferred to a differential equation in the flow $\dot{x}$, yielding the behaviour

$$\mathcal{B}_p = \left\{(x, F_e, F_c) : \mathbb{T} \mapsto \mathbb{W} \middle| F_e - m\ddot{x} - cx + F_c = 0\right\} \tag{4.8}$$

on a continuous time axis $\mathbb{T} = \mathbb{R}$ and a signal space $\mathbb{W}_p = (x, F_e, F_c) \subseteq \mathbb{R}^3$.

This system exhibits sustained oscillation, which poses a problem in numerous applications. A reasonable control target is to reach a damped behaviour in order to have a decreasing amplitude of oscillation. This introduced dissipativity increases the stability of systems, therefore dampers, often as simple as friction dampers, are included in many designs.

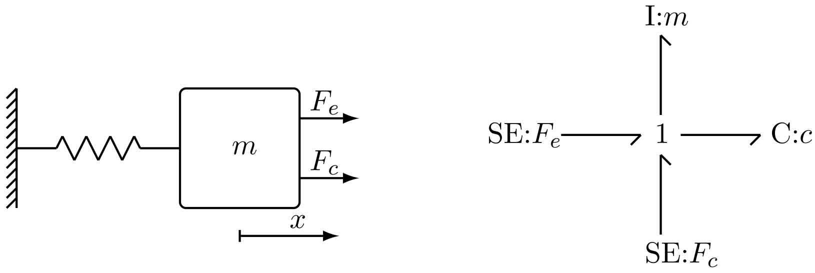
A damping element R with damping coefficient b, as depicted in Figure 4.15(c), improves this behaviour depending on the damping coefficient. The behaviour of the controller is

$$\mathcal{B}_c = \left\{(x, F_c) : \mathbb{T} \mapsto \mathbb{W}_c \middle| F_c + b\dot{x} = 0\right\} \tag{4.9}$$

using a common time axis $\mathbb{T}$ with the plant and a controller signal space $\mathbb{W}_p = (x, F_c) \subseteq \mathbb{R}^2$.

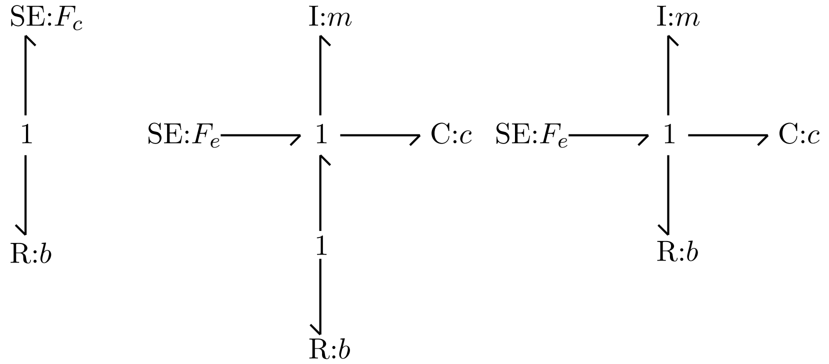
The controller can be fixed to the mass and the relevant mechanical basis (drawn as a wall in Figure 4.15(a)), this means the controller is interconnected on both its available ports. The port F_c is no longer a manifest variable after interconnection, while the position variable x remains an external, i.e. manifest, variable. The interconnection does not make use of the absolute position, only the velocity of the mass, the flow $\dot{x}$ is of relevance for the control force F_c, which is an effort in the mechanical translational domain. For these reasons, the junction to interconnect the controller is a 1-junction, as shown in Figure 4.15(d).

The second 1-junction in Figure 4.15(d) is redundant due to the associative property of the underlying additions and can be omitted. In this case the generalised resistor R:b representing the damper is directly connected to the initial 1-junction of the mass-spring system, yielding the simplified version given in Figure 4.15(e).



(a) Mass-spring system

(b) Mass-spring system as a bond graph



(c) Control system submodel as a bond graph

(d) Controlled mass-spring system

(e) Controlled mass-spring system simplified

Figure 4.15: System ((a),(b)) and interconnected controller (c) as bond graphs in full (d) and simplified (e) form

The manifest behaviour of the controlled system can be gained from the bond graph as

$$\mathcal{B} = \left\{(x, F_e) : \mathbb{T} \mapsto \mathbb{W} \middle| F_e - m\ddot{x} - cx - b\dot{x} = 0\right\} \tag{4.10}$$

for the initial time axis $\mathbb{T}$ and the signal space $\mathbb{W} = (x, F_e) \subseteq \mathbb{R}^2$, which is equivalent to the result obtained in Example 6. The different procedure of obtaining this result however leads to it being achieved without stating the full behaviour first.

The above example indicates that while system representations as physical abstraction are very convenient for acausal modelling, since they reflect the true system setup, for support of analytical calculations bond graphs are more appropriate as they reduce the amount of calculations to a minimum. Further, similar to block diagram algebra, the required operations are standardised and thus simple and little error prone. This advantage of bond graphs for analytical processes comes at the cost of a higher level of abstraction, which sometimes requires an extra step in the analysis.

4.4 Application study

4.4.1 Motivation

The use of rapid prototyping tools is known to provide an improvement in terms of cost and time in the development of systems. In order to be able to offer competitive system designs, it is on the one hand necessary to know about the technical limits of the equipment, and on the other hand it is necessary to obtain the load on the system by simulation.

In the present case, the rapid prototyping of brake equipment for railway vehicles, the choice is among several materials for brake discs and pads, each of them having certain advantages and drawbacks, partly determined by the thermal load on the component. In contrast to automotive systems, the route profile of a railway vehicle is frequently known at the time of design, thus the brake system is tailored to the specific needs of this route profile by the help of expensive dynamometer tests, which may be reduced by help of an improved model in conjunction with an optimised test profile.

In the tender and early design stages, instead of dynamometer tests, thermal simulations using specialised software are performed. Driven by higher requirements on performance and documentation, customers require an increasing number of influences to be considered in simulations and dynamometer tests, such as brake cylinder filling time, different blending concepts, ride resistances and deteriorated scenarios. Currently, such features are included in custom programmed software yielding an appropriate usability but difficult extendability to encompass new features. For this purpose it may be helpful to incorporate the thermal simulation kernel into a block diagram simulation software such as Simulink or Dymola in order to make the simulation environment more flexible and easily adaptable to system oriented needs. In order to accomplish this and to make more detailed simulations feasible in terms of computational load, the partial differen-

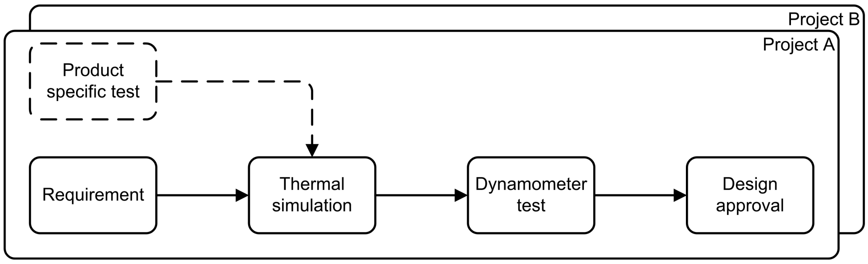


Figure 4.16: Classical way of performance approval per project

tial equation model should be reduced to an ordinary differential equation model, possibly of nonlinear type.

Further motivation for this work stems from the increasing numbers of so called *platform vehicles*. These vehicles are developed keeping in view the needs of the market, and in contrast to former developments, at the time of design, no operator and especially no route profile is known. To support this new way of railway vehicle development, improved grey-box modelling may help to predict performances on certain route profiles based on measurements on different route profiles or even synthetic profiles.

The classic way for the type approval of brake systems is depicted in Figure 4.16, while the more platform oriented approach is depicted in Figure 4.17. From a comparison it becomes obvious that as soon as a sufficient amount of performance data is recorded, further costly dynamometer test may be avoided, given that a reliable, well tuned grey box model substitutes these. It is reasonable to assume that an appropriate calibration profile as compared to a mission simulation based on route data of an existing railway track will prove advantageous for tuning this model.

4.4.2 Disc brake system

In railway braking, several types of brake discs, made from a variety of materials are used. They are commonly distinguished by the way of mounting, discs may be either mounted to wheel or axle of the vehicle. The latter type is available in ventilated or non-ventilated form and is made from various types of cast iron or steel. For future demands, other materials are investigated, ranging from aluminium to ceramic materials. The brake pads are categorized into organic and sinter material, with a wealth of mixtures being offered by the companies in the market.

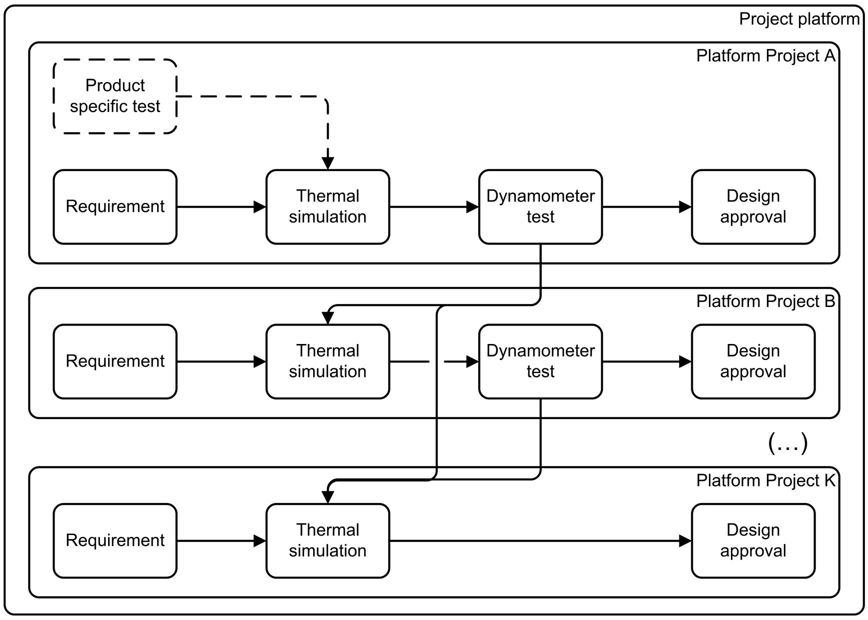


Figure 4.17: Platform oriented way of performance approval

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Figure 4.18: Ventilated axle mounted brake discs (source: www.faiveleytransport.com)

To achieve a braking force, the pads are pressed to the discs by a brake calliper powered by a pneumatic or hydraulic actuator. A torque is consequently applied to the rotating disc, decelerating the train.

It is generally agreed that the work produced in braking is mostly converted to heat (Sheridan *et al.*, 1988), increasing brake and pad temperatures during the braking process. This heat limits the performance of the braking system, as in addition to the thermal load large forces are generated (Dufrenoy, 2004).

4.4.3 Analytical modelling

Preliminaries

The model is derived from physical principles and is limited to one spatial dimension, as the search for the maximum temperatures faced in braking is a one-dimensional problem as shown in (Dunaevsky, 1991). In addition to the heat conduction in the brake disc, the heat transfer between brake disc and brake pad is modelled, as the temperatures of both friction partners have to be considered. The modelling of the pad temperature is novel in the context of models for mission simulation, as this temperature is frequently assumed to be in the range of the disc surface temperature. This assumption is shown not to be valid under all conditions, as reported by (Siroux *et al.*, 2008) and (Dufrenoy, 2004), therefore the model of heat flux distribution proposed in (Vernersson, 2007) for tread brakes is assumed here.

In order to save computation time and not to attempt to model the aerodynamic effects too closely in order to keep tuning simple, the symmetry of the arrangement of a brake disc and pads is exploited for simplification of the model. This symmetrical arrangement is shown in Figure 4.19, the model only covers the left part excluding the cooling part.

Pad and disc model

As discussed above, the problem may be reduced to a one-dimensional partial differential equation problem, as the angular thermal differences are found to be negligible in (Dufrenoy, 2004), whereas the radial distributions are omitted in this study as the resulting temperature is required in scalar form. The problem under consideration is thus governed by the

Figure 4.19: Disc brake arrangement

one-dimensional heat equation

$$\rho(x)c(x)\frac{\partial\theta}{\partial t} = \lambda(x)\frac{\partial^2\theta}{\partial x^2} \tag{4.11}$$

where θ denotes the temperature, t time, x the axial position, λ the heat conduction coefficient, c the heat capacity and ρ the density of the material in this position x. The material constants are different between disc and pad, thus the parameters are considered as spatial varying parameters, but are assumed to be independent of temperature and time.

The second-order partial differential equation (4.11) is defined on a bounded set $D = [0, t_{max}] \times [x_0, x_{max}] \subset \mathbb{R}^2$. As an initial condition we assume the disc to be cooled to ambient temperature

$$\theta(0, x) = \theta_{amb} \quad \forall x \in D \tag{4.12}$$

The cooling is mainly achieved by convection on both disc and pad surfaces under consideration, the cooling performance of the external surfaces is thus determined by the Robin boundary conditions (Oberst, 1990)

$$\lambda\left(\frac{\partial\theta}{\partial x}\right)_{x_0} = \alpha_{x_0}\left(\theta_{x_0} - \theta_{ambient}\right) \tag{4.13}$$

$$\lambda\left(\frac{\partial\theta}{\partial x}\right)_{x_{max}} = \alpha_{x_{max}}\left(\theta_{x_{max}} - \theta_{ambient}\right)\left(\frac{v(t)v_{max}}{2v_{max}}\right) \tag{4.14}$$

with the two convection coefficients α_{x_0} and $\alpha_{x_{max}}$ being the convection coefficients of the pad and disc surface, respectively. The velocity dependent term in (4.14) approximates the ventilation effect of the inside of the brake disc by assuming a linear dependence between heat dissipation and velocity.

The thermal power introduced during stop braking, i.e. braking to reduce velocity to a full stop, omitting potential energy introduced by slopes, is given by

$$P(t) = -\frac{\partial}{\partial t}\frac{1}{2}m\left(v(t)\right)^2 = -mv(t)a(t) \tag{4.15}$$

under the reasonable assumption of a constant mass during the braking process. In the model presented in (Vernersson, 2007), the heat flux consequent to the braking process is introduced into the boundary layer forming between brake pad and disc. This boundary layer is assumed to form a higher thermal resistance than the neighbouring bodies, governing the distribution of heat energy between the two. The dissipation of energy by convection takes place by the same mechanisms as in (4.13) and (4.14), with a different convection coefficient due to a different surface shape and quality. Depending on the state of the brake (cooling or braking) this yields a source or sink of energy between the pad and disc governed by

$$\lambda\left(\frac{\partial\theta}{\partial x}\right)_{x_{boundary}} = \alpha_{x_{bounday}}\left(\theta_{x_{boundary}} - \theta_{ambient}\right) - q(t) \tag{4.16}$$

where $q(t)$ is the heat flux density introduced by braking relative to the area in the friction contact. Obviously, the resulting layer is assumed to be of infinitely small thickness.

Together with the heat equation (4.11), it is possible to express this setup in the form of a bond graph, see Figure 4.20. In this bond graph, the introduction of a heat flux is depicted in the form of a source of flow (SF) and the dissipation of heat at both boundaries by the respective resistors (R). In case of heat dissipation via the disc surfaces in the absence of brake energy introduces into the disc, the source is considered negative, i.e. as a sink. The inertia (I) and capacitor (C) components form a discretised spatial section of the disc-boundary-pad setup.

Formalising the behavioural equations

Throughout the analytical development of the model equations in the preceding chapters, no causal direction was introduced. Especially thermal systems, partly due to their slow dynamics, leading to a possible interpretation as a low propagation velocity of heat, are very sensitive to an *a priori*

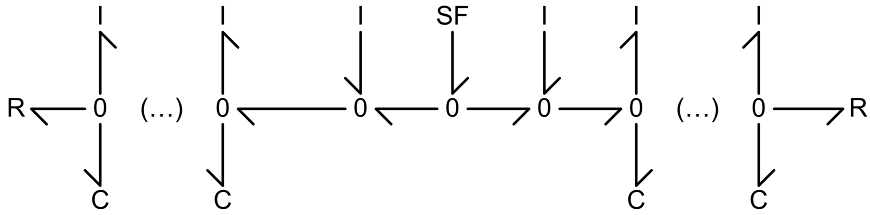


Figure 4.20: Disc-boundary-pad setup as bond graph

choice of the input and output variables.

In order to illustrate one of the problems that may arise from a too early distinction into input and output, consider an infinitely small block of matter, governed by the heat equation (4.11). It is known that the dependence of heat flux density q and temperature θ is described by the simple proportional law $q \propto \theta$. Under the assumption that the heat flux is caused by temperature, one may say that the temperature is an input variable and the heat flux is the output variable. In order to assemble the brake disc under consideration, it is necessary to connect these small blocks. This interconnection will then invert the input-output relation derived above, as the second interconnected block is heated by the heat flux, i.e. it's output now becomes an input. For this reason, the acausal notions put forward in the BF are adhered to and the behavioural equations are formalised.

The partial differential equation problem under consideration has a two-dimensional time axis

$$\mathbb{T} = [0, t_{max}] \times [x_0, x_{max}] \subset \mathbb{R}^2$$

as well as a two-dimensional signal space

$$\mathbb{W} = (\theta, q) \subseteq \mathbb{R}^+ \times \mathbb{R}$$

The behaviour is governed by the heat equation and the appropriate boundary conditions, (4.11), (4.13), (4.14) and (4.16), yielding

$$\begin{aligned}\mathcal{B} = \Big\{ &(\theta, q) : \mathbb{T} \mapsto \mathbb{W} \big| \rho c \frac{\partial \theta}{\partial t} = \lambda \frac{\partial^2 \theta}{\partial x^2} \wedge \\ &\lambda \left(\frac{\partial \theta}{\partial x} \right)_{x_0} = \alpha_{x_0} \left(\theta_{x_0} - \theta_{ambient} \right) \wedge \\ &\lambda \left(\frac{\partial \theta}{\partial x} \right)_{x_{max}} = \alpha_{x_{max}} \left(\theta_{x_{max}} - \theta_{ambient} \right) \wedge \\ &\lambda \left(\frac{\partial \theta}{\partial x} \right)_{x_{boundary}} = \alpha_{x_{bounday}} \left(\theta_{x_{boundary}} - \theta_{ambient} \right) - q(t) \Big\}\end{aligned}$$

This formulation has the advantage of being formalised without any *a priori* notions of causal direction of the system. Due to its partial derivative nature and the set membership definition, it is not directly suitable for simulation, the necessary refinements will be described in the next section.

The separation of modelling and simulation however achieved in this way leads to a well reusable and well documentable model. The documentation of the model is especially important since its results will be used for type approval in later stages and thus the quality of documentation will partly decide about the usefulness of the model.

Calibration and Simulation

Despite the continuous formulation in space and time in (4.11), for a simulation it is common to discretise at least the spatial coordinates. The simulation is achieved by help of the MATLAB built-in function `pdepe`, which uses a discrete spatial axis and continuous time axis. To this function, the differential equation as well as the boundary conditions are defined as continuous time and space functions, a discretisation is executed with a specified mesh. In the simulations presented the mesh has a resolution of 0.2 mm, the thickness of the overall setup is 66 mm. As this discretisation does not enable the user to have an infinitely small boundary layer, a thickness of five space samples is assumed.

While the material constants contained in (4.11) are well known, refer to (Saumweber, 1969), the parameters in (4.13), (4.14) and (4.16) are not easy to measure and not well studied in comparison the material constants. For this reason, a grey box approach is chosen where the material constants are taken from literature, whereas the convection coefficients are manually tuned

Property	Variable	Value	Tuned
Thermal diffusivity pad	$\alpha\|_{x\in X_{pad}}$	$4.63\cdot 10^{-5}\frac{\mathrm{m}^2}{s}$	No
Thermal diffusivity boundary	$\alpha\|_{x\in X_{boundary}}$	$2\cdot 10^{-6}\frac{\mathrm{m}^2}{s}$	No
Thermal diffusivity	$\alpha\|_{x\in X_{disc}}$	$1.11\cdot 10^{-5}\frac{\mathrm{m}^2}{s}$	No
Ambient temperature	θ_{amb}	323 K	No
Maximum velocity	v_{max}	160 km/h	No
Mass	m	12150 kg	No
Acceleration	a	1.65 $\frac{\mathrm{m}}{\mathrm{s}^2}$	No
Pad width	X_{pad}	[0, 35] mm	No
Boundary layer	$X_{boundary}$	]35, 36[mm	No
Disc width	X_{disc}	[36,63] mm	No
Convection coefficient pad	α_{x_0}	$4.6\cdot 10^{-15}$	Yes
Convection coefficient boundary	$\alpha_{x_{boundary}}$	$5\cdot 10^{-12}$	Yes
Convection coeficient disc	$\alpha_{x_{max}}$	$1\cdot 10^{-16}$	Yes

Table 4.3: Parameter set for simulation of thermal behaviour of a disc brake, manually tuned parameters are marked 'Yes' in the final column

based on an existing software solution as no suitable test bench results are present to date. This software models the inner and outer disc temperature and is generally in good accordance with dynamometer tests.

The brake disc in the software simulation is a nodular grey cast iron brake disc, used with an organic high temperature brake pad. The material constants can be simplified to a single factor $a(x) = \frac{\lambda(x)}{\rho(x)c(x)}$, the thermal diffusivity. The values of the material constants are varying with temperature, however for the simulation constant values are used. The parameters for the simulation are given in Table 4.3.

A comparison between the existing software, indices denoted with *true* for inner and outer disc temperature, and the additional result of the estimation of the pad temperature is shown in Figure 4.21. Here a good accordance can be found, especially in the maxima and the temperature after cooling. The qualitative shape of the curves deteriorates slightly, partly as the PDE model uses a finer grid, partly due to the nonlinear nature of the problem approximated by a linear time invariant PDE.

The pseudocolour image in Figure 4.22 shows the local temperature evolving over time, here it is interesting to observe that while the disc takes time to be heated through, the pad has a higher thermal conductivity and thus has a quite low spatial temperature gradient. Considering the comparably low mechanical strength of brake pad materials, this choice together

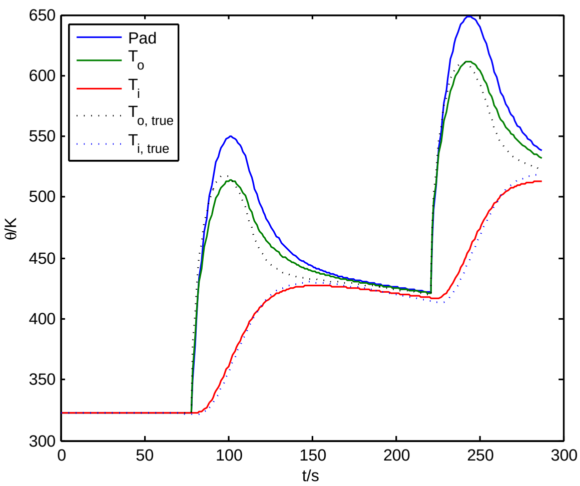


Figure 4.21: Inner and outer disc and pad temperature

with the lower convection coefficient reduces thermal stresses of the pad. The brake disc material, a special globular cast iron, is able to bear high thermal stress, thus it can offer higher convection cooling to the overall system.

These desired time and space evolutions of the temperature in the different materials indeed exhibit all features of a control system in the BF. The control target is to cool the system as good as possible while not introducing unbearable thermal stresses to the system. The parameters that can be influenced in this tuning process are the convection coefficients and the thermal diffusivities of the materials of brake disc and pad as well as the ventilation of the brake disc.

4.5 Concluding remarks

Abstraction and interpretability are considered from a behavioural perspective, indicating that the validity of models has to be chosen appropriately for the intended interpretation in the sense of modelling for purpose. The model classes suitable for application form a set-subset structure, the validity of models can be extended by applying model structures from the next higher level of superset. This means that in particular the validity of a model for systems that exchange information and power bidirectionally

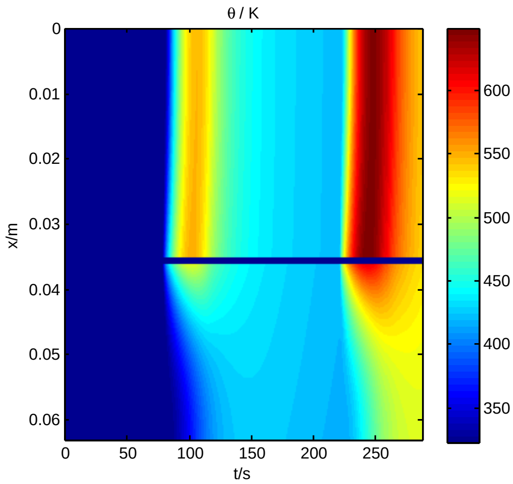


Figure 4.22: Temperature profile

can be obtained by using acausal models as expressed in the Behavioural Framework.

Typically a first step in abstraction of a system is taken by transforming the system into an abstract graphical representation. Three frameworks for these graphical representation are presented, these are block diagrams, bond graphs and physical abstractions. All three show different advantages and drawbacks, however the block diagram is applied very frequently despite the fact that it introduces an input/output structure *a priori*, making it inappropriate for use in conjunction with the BF. As a novelty to the field, links between model formulations in the BF and graphical model representations are established. These links enable the user to apply the BF with and without software assistance.

Two examples are visited, these are a model of the longitudinal dynamics of a train set and an example of controller design based on control by interconnection. The former exhibits the relative superiority of acausal modelling and simulation over block diagram modelling both in simplicity and interpretability by detecting undesired behaviour that may remain unnoticed when simulated in block diagram tools. The latter example is executed analytically and shows that despite their high level of abstraction, bond graphs are well suited for analytical processing of models for control and allow a modular approach. In addition to this good applicability to control problems, it is found that when using bond graphs, it is possible to express the manifest behaviour without first stating the full behaviour and later simplifying the overall structure.

As an application study, a disc brake arrangement for a train set is discussed. In the context of technical applications, first principles models are a very powerful means to shorten development cycles and make full use of platform developments.

A linear time invariant PDE model of the disc and pad setup is developed analytically. As the distinction between input and output is not obvious in this problem, the problem is considered from a behavioural point of view and a dynamical system is formulated in terms of behavioural equations is presented. After manual tuning of the convection coefficients, the model is in good agreement with the existing software solution in terms of maximum and limit temperatures. Also the slightly higher pad temperature appears reasonable.

The cooling of the brake disc and pad is considered as a control system following the control by interconnection paradigm. It becomes obvious that not only the cooling performance but also the temporal and spatial evolution of the temperatures forms an important performance criterion. This control system cannot be considered when the view is restricted to the intelligent control scheme, thus reducing potential for optimisation.

The research presented in this section can be extended by further applications, particularly to nonlinear system. The resulting models can exhibit their usefulness best when they are calibrated on measured data. The links to graphical model representations presented in the present chapter is tied for analytical models and empirically designed control systems. A formal connection to control design techniques in the BF will further increase the usefulness of these links.

Chapter 5

Extension of the validity of models

And as every present state of a simple substance
is naturally a consequence of its preceding state,
so its present is pregnant with its future.

Gottfried Wilhelm von Leibniz

5.1 Introduction

In the previous chapter, the validity of models was increased by encompassing acausal linear time invariant models in addition to their causal counterparts. This, together with the appropriate graphical representations, shows some advantages for systems that reverse their power flow direction during operation.

A further limitation of model validity is the restriction of the model class to encompass only linear models. Especially in the behavioural framework, this restriction is frequently accepted, as despite its general formulations, only few tools and model structures have been examined in literature. As with the very global nonlinear model class in the input/output framework, the class of acausal nonlinear models is the most general in this framework.

This generality is difficult to handle in practical applications, for this purpose certain frequently applied subclasses are presented. These subclasses each show a different set of features, all exhibit behaviours that the class of linear models is lacking. A particular focus is put on the the application of bilinear modelling techniques, as this class turns out to be appropriate for acausal nonlinear modelling.

An acausal bilinear model representation, the so-called bilinear extended kernel representation, is introduced in both discrete and continuous time formulation. The existence and uniqueness of solutions to both discrete and continuous time forms is investigated, it is shown that unique solutions exist almost everywhere. The representation of a bilinear system as a bond graph leads to additional insight, as in this energy conserving form the exothermic or endothermic nature is made obvious.

The novel model structure is applied to simulation and empirical modelling, the latter being based on data of nonlinear models resembling practical systems closely or recorded data from system experiments.

5.2 Nonlinear systems and models

This work aims to evaluate the applicability of the BF to practical systems. One of the key properties that distinguish practical systems from their idealised counterparts are nonlinearities. These nonlinearities appear in their most obvious form as time or state dependent parameters of system components or saturation due to limited power available in the system and may range to highly nonlinear structures in complex chemical systems. For this reason, a study of the applicability of any theoretical framework must stand the test against nonlinearities in modelling, system identification and control.

A literature review of nonlinear modelling techniques was presented in Section 3.3, while here subclasses of practical relevance are introduced.

Within the class of nonlinear systems, some subclasses stand out due to their simple structure and widely recognised applicability, among these are Wiener and Hammerstein systems as well as bilinear systems.

The former are constructed by concatenation of linear dynamic system and a static nonlinearity while the latter represent a first order series expansion of the general NARMAX model (Billings and Voon, 1986)

$$\begin{aligned} y(k) =& F\left(y(k-1), y(k-2), \cdots, y(k-p), u(k), u(k-1), \cdots, u(k-q),\right. \\ & \left. e(k-1), e(k-2), \cdots, e(k-r)\right) + e(k) \end{aligned} \tag{5.1}$$

These model classes formed by restriction of the NARMAX model class are introduced in detail in the following sections. Their qualitative behaviour is summarised and compared in Section 5.2.4.

5.2.1 Bilinear systems

In many first principles modelling tasks, a restriction of (5.1) occurs naturally. This restriction exhibits a linear dependence on input and state when changed separately, but when the variables are changed together, the systems do not behave linearly. Drawing a parallel to the class of bilinear maps in contrast to linear maps, this class of systems is termed bilinear systems by Mohler (Mohler, 1973).

Mohlers initial study was led in the context of modelling of nuclear fissions. Assuming a nuclear fission reaction of ^{235}U as in (Mohler and Shen, 1970), the change in neutron population n can be expressed as

$$\frac{dn}{dt} = \left(\tfrac{k(1-\beta)-1}{l}\right) n + \textstyle\sum_{i=1}^{k} \lambda_i c_i + s \tag{5.2}$$

$$\frac{dc_i}{dt} = k\tfrac{\beta_i}{l} n - \lambda_i c_i, \quad i = 1, \cdots, 6 \tag{5.3}$$

with k the number of prompt offsprings, l the mean prompt neutron generation time, s the amount of neutrons stemming from the neutron source and β the portion of delayed neutrons. Equation (5.2) also incorporates fission caused by delayed neutron sources, ^{235}U generates six groups of delayed neutron sources, so-called precursors, with decay rates λ_i and population c_i.

The states of the system are the number of neutrons n in the reactor and the number of precursors c_i. The external variables accessible for control are s as an additive neutron source and k as a multiplication constant, reflecting the average number of prompt neutron offsprings. While the former is an additive control as present in linear model structures, the latter is acting on the system multiplied by the state n.

A continuous time representation of a bilinear system

$$\frac{dx}{dt} = Ax + \sum_{k=1}^{m} B_k u_k x + Cu \tag{5.4}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ an input vector and the matrices $A \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times n} \forall k \in [1, \cdots, m]$ and $C \in \mathbb{R}^{n\times m}$ are constant matrices describing the system.

The related discrete time representation of a bilinear system is (Dunoyer *et al.*, 1997)

$$\begin{aligned} y(k) = &\sum_{i=1}^{n_a} a_i y(k-i) + \sum_{i=0}^{n_b} b_i y(k-\nu-i) \\ &+ \sum_{j=1}^{n_a}\sum_{i=1}^{n_b} \eta_{ij} u(k-\nu-i+1) y(k-\nu-j) \end{aligned} \tag{5.5}$$

with the integer time index k and a delay $\nu \geq 1$. Bilinear systems have been found to be appropriate for modelling of thermal systems and chemical reactions of exothermic and endothermic type as well as certain mechanical systems (Larkowski *et al.*, 2012; Martineau *et al.*, 2004).

For graphical modelling of bilinear systems, mostly block diagrams are employed. The common block diagram of a first order SISO bilinear system is shown in Figure 5.1.

The formulation of a bilinear system as in (5.5) is causal since it declares $y(k)$ to be depending on past values of u, thus, when (5.5) is solved for u, u depends on past and future values of y. Therefore a revision of bilin-

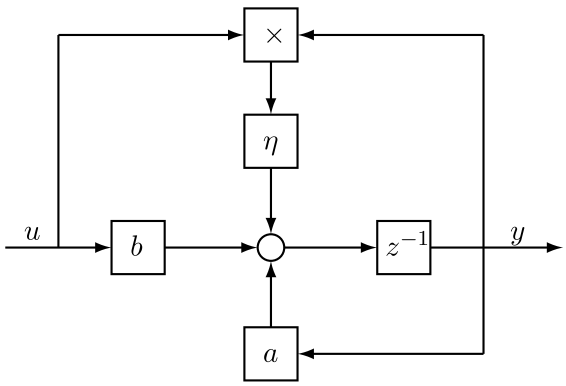


Figure 5.1: Block diagram representation of a discrete time first order bilinear system

ear systems from a behavioural perspective may help to integrate bilinear systems fully into the behavioural framework and in this way, bilinear and behavioural approaches can be combined.

5.2.2 Hammerstein systems

The class of Hammerstein systems belongs to the class of block oriented systems and consists of the concatenation of a static nonlinearity and a dynamic system (Pearson, 1995). The static nonlinearity acts on the input of the dynamic system, as depicted in Figure 5.2. This structure is quite popular in literature and applications, since it can approximate a multitude of technical systems. As an example for the intuitive appropriateness of a Hammerstein system, consider an approximately linear system in the form of a rotational mass with negligible friction driven by an actuator providing finite torque or a saturation at a given frequency.

Hammerstein systems, depending on the nature of the nonlinearity, may exhibit input multiplicities (Pearson and Pottmann, 2000). This is the case when the static nonlinearity cannot be uniquely inverted, which results in a causal direction assumed *a priori*.

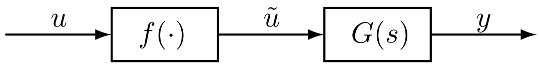


Figure 5.2: Block diagram representation of a Hammerstein system

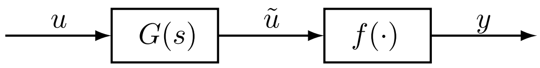


Figure 5.3: Block diagram representation of a Wiener system

5.2.3 Wiener systems

Changing the order of the static nonlinearity and the linear dynamic system of a Hammerstein system results in a Wiener system. Wiener systems are thus made up of a dynamic linear system followed by a static nonlinearity, as depicted in Figure 5.3. As for Hammerstein systems, Wiener systems are popular in control engineering practice as well as theory due to their appealing intuitivity. As an example, consider the actuator of the above example exhibiting a finite torque. Detailing this actuator further will perhaps exhibit that its dynamic behaviour can be approximated sufficiently by a linear system and only the saturation needs to be expressed statically, thus the static nonlinearity following the linear dynamic system suffices.

When a non-invertible nonlinearity is chosen for $f(\cdot)$, Wiener systems exhibit output multiplicities.

5.2.4 Properties of certain classes of nonlinear systems

Nonlinear systems frequently exhibit behaviours that are incompatible with the linearity of the model employed to explain the system. In order to

Model Structure	Input Multiplicity	Output Multiplicity	Subharmonic Generation	Existence of Inverse
NARMAX	Yes	Yes	Yes	Partly
Hammerstein	Yes	No	No	Partly
Wiener	No	Yes	No	Partly
Bilinear	No	No	Yes	Full
Feedback	No	Yes	No	Non

Table 5.1: Overview over some particularities of certain classes of nonlinear models

receive a model valid also for these nonlinear behaviours, it is necessary to choose a model structure that is able to explain the observed phenomena. The behaviour particularities considered here include harmonic generation, jump phenomena, synchronization, chaos and amplitude dependent responses, such as output multiplicities (Pearson, 1995). These phenomena discussed here are explained in Section 3.3.

If a nonlinear model structure is selected for application in the BF, also the existence of the inverse of the model equations may be considered as a criterion. Consequently, input or output multiplicities inhibit the existence of the inverse of the behavioural equations. Nonlinear systems exhibiting input or output multiplicities are encompassed in the BF, however model structures exhibiting such behaviour may not be suitable for all applications. Behavioural equations that cannot be inverted are not suitable for multidirectional simulation, i.e. simulation in all causal directions allowed by the variables defined.

Table 5.1, compiled from (Pearson, 2003, 1995; Pearson and Pottmann, 2000), lists the known properties for some frequently discussed classes of nonlinear models. It should be noted that Bilinear Systems exhibit subharmonics only under extreme conditions (Pearson, 1995). Further to this information on the nonlinear behaviour, the existence of the inverse model structure is shown in this table, displayed as three levels 'non', 'partly' or 'full' existence of the inverse model.

The analysis of the available model structures puts the class of bilinear models into a favourable position, since this is the only structure that can be inverted uniquely in all cases, except for a finite number of points in the operating range. It is further possible to use Wiener or Hammerstein models with invertible nonlinear functions and acausal dynamic blocks. Due to their advantage of being invertible in all cases, the bilinear model structure is applied in the sequel of this chapter, with an initial step being the representation of bilinear models in the BF.

5.3 Bilinear systems in the Behavioural Framework

5.3.1 Introduction

The class of bilinear models is the only model class that can inverted uniquely in all cases, for this purpose the representation of bilinear models in the BF is expected to improve the applicability of the BF to practical systems. The representation is achieved by making use of an acausal interconnection of a static nonlinear function to a dynamic plant.

In this way, the dynamic part of the behaviour remains in the domain of the well developed BF, while the nonlinear function can be treated separately. This approach applies techniques present in the BF, namely interconnection and latent variables, together with an algebraic bilinear form.

The system investigated in this section is a so-called diagonal bilinear model, as it only contains products of variables with the same lag in the discrete time case. The diagonal bilinear model subclass is a well applicable, mostly sufficient model class. The approach is not limited to this diagonal structure, as by adaptation of the dynamic part and the latent variables, off-diagonal entries can be generated.

In the continuous time case, the structure does not exclusively contain products of variables with the same degree of differential, this is due to product rule of differentiation.

5.3.2 Representation of bilinear systems in the Behavioural Framework

The development of a representation of bilinear systems in the BF will be carried out in a two free variable setting, which is the conceptual equivalent of a single-input-single-output system in the input/output paradigm. All steps generalise to the case of more than two free variables.

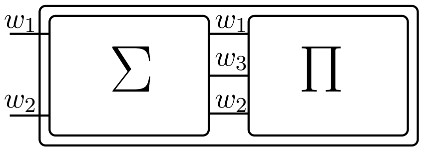


Figure 5.4: Interconnection between dynamic system and static nonlinearity

The system that serves as a basis for development is a dynamic system with a discrete or continuous time axis, $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, and a two dimensional manifest signal space $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \subseteq \mathbb{R}^2$. The behaviour of the system will be developed in the following steps.

As a first step, it is necessary to add latent variables to the signal space, these latent variables serve for the exchange of the bilinear term between the static nonlinearity and the dynamic system. As this latent variable has to contain the result of the bilinear form

$$f : \mathbb{W}_1 \times \mathbb{W}_2 \to \mathbb{W}_3, \quad f(w_1, w_2) := w_1 w_2 \tag{5.6}$$

the signal space of this additional latent variable has to be

$$\{w_1 w_2 \; \forall w_1 \in \mathbb{W}_1 \wedge w_2 \in \mathbb{W}_2\} \subseteq \mathbb{R}$$

The bilinear extended signal space containing latent variables is thus given by

$$\mathbb{W}_{ext} = \mathbb{W}_1 \times \mathbb{W}_2 \times \{w_1 w_2 \, | w_1 \in \mathbb{W}_1 \wedge w_2 \in \mathbb{W}_2\} \subseteq \mathbb{R}^3 \tag{5.7}$$

As a next step, it is required to interconnect the bilinear form, as a static nonlinear element, to the dynamic system. The static nonlinearity Π shares the time axis $\mathbb{T}$ with the dynamic system Σ, has the signal space as defined in (5.7) and a behaviour

$$\mathcal{B}_\Pi = \left\{ (w_1, w_2, w_3)^T \in \mathbb{W}_{ext} | w_3 = w_1 w_2 \right\} \tag{5.8}$$

The dynamical system Σ is assumed to be expressed in terms of a kernel representation, leading to a full behaviour

$$\mathcal{B}_\Sigma = \left\{ (w_1, w_2, w_3)^T \in \mathbb{W}_{ext}^{\mathbb{T}} | R_{full} \left(w_1, w_2, w_3 \right)^T = 0 \right\} \tag{5.9}$$

for a polynomial matrix

$$R_{full}(\sigma) = (R_1(\triangle) \quad R_2(\triangle) \quad R_3(\triangle)) \in \mathbb{R}^{l\times 3}[\triangle] \tag{5.10}$$

over the differential or shift operator $\triangle$ for an order or lag l system.

The interconnection along the ports w_1, w_2 and w_3 as in Definition 12 leads to the overall full behaviour

$$\begin{aligned} \mathcal{B}_f &= \mathcal{B}_\Sigma \cap \mathcal{B}_\Pi \\ &= \left\{ (w_1, w_2, w_3)^T \in \mathbb{W}_{ext}^{\mathbb{T}} | R_{full} \left(w_1, w_2, w_3 \right)^T = 0 \wedge w_3 = w_1 w_2 \right\} \end{aligned} \tag{5.11}$$

The manifest behaviour can be derived from (5.11) by substituting $w_3 = w_1 w_2$ in the kernel representation of the full behaviour, yielding the manifest behaviour

$$\mathcal{B} = \left\{ (w_1, w_2)^T \in \mathbb{W}_{ext}^{\mathbb{T}} | R_1(\triangle) w_1 + R_2(\triangle) w_2 + R_3(\triangle) w_1 w_2 = 0 \right\} \tag{5.12}$$

5.3.3 Existence and uniqueness of solutions

Having established the formulation of bilinear systems in the BF, it is natural to ask for the existence and especially uniqueness of the solutions. It is necessary to consider the cases of discrete and continuous time axis separately.

As in the development of the representation above, the treatment will be limited to case of two external variables. The possible extension will be discussed in the individual case.

Discrete time case

In the discrete time case, it suffices to solve the discrete time kernel representation for both variables. For this purpose, it is useful to introduce an auxiliary polynomial

$$\tilde{R}_i(\sigma) = R_i(\sigma) - R_{i,0} = R_{i,1}\sigma + \ldots + R_{i,l}\sigma^l \tag{5.13}$$

for coefficients of $R_i(\sigma)$ termed $R_{i,j}$.

It is then possible to express the kernel representation in (5.12),

$$R_1(\sigma)w_1 + R_2(\sigma)w_2 + R_3(\sigma)w_1w_2 = 0,$$

as

$$R_{1,0}w_1 + R_{2,0}w_2 + R_{3,0}w_1w_2 + \tilde{R}_1(\sigma)w_1 + \tilde{R}_2(\sigma)w_2 + \tilde{R}_3(\sigma)w_1w_2 = 0 \quad (5.14)$$

For the diagonal bilinear system under consideration, the summands containing $\tilde{R}_i$ only contain lagged terms of w_i and are therefore not required for calculation of the current variable w_i. It is therefore possible to solve (5.14) for w_1 or w_2, depending on the causal direction of the system. This results in

$$w_1 = \frac{-\left(\tilde{R}_1(\sigma)w_1 + \tilde{R}_2(\sigma)w_2 + \tilde{R}_3(\sigma)w_1w_2 + R_{2,0}w_2\right)}{R_{1,0} + R_{3,0}w_2} \quad (5.15)$$

for $w_2 \neq \frac{-R_{1,0}}{R_{3,0}}$ and

$$w_2 = \frac{-\left(\tilde{R}_1(\sigma)w_1 + \tilde{R}_2(\sigma)w_2 + \tilde{R}_3(\sigma)w_1w_2 + R_{1,0}w_1\right)}{R_{2,0} + R_{3,0}w_1} \quad (5.16)$$

for $w_1 \neq \frac{-R_{2,0}}{R_{3,0}}$.

Solutions therefore exist and are unique for all points except $w_2 \neq \frac{-R_{1,0}}{R_{3,0}}$ and $w_1 \neq \frac{-R_{2,0}}{R_{3,0}}$. Since the value of the variables grows to infinity left and right of this point, this points coincide with points of state dependent instability. When the above treatment is generalised to q variables, up to $q-1$ of these points exist. Furthermore, the introduction of off-diagonal bilinear terms, i.e. bilinear terms that do not have the same lag, may increase the number of points where the solution is not unique further. For each off-diagonal bilinear having a factor with lag 0, one further point of nonuniqeness is generated.

Continuous time case

For the continuous time case, it suffices to show the existence and uniqueness for first order bilinear systems. In the case of a higher order, a transformation according to (2.10) can be applied.

The first order bilinear extended kernel representation can be written explicitly as

$$\left(R_{1,0} + R_{1,1}\frac{d}{dt}\right) w_1 + \left(R_{2,0} + R_{2,1}\frac{d}{dt}\right) w_2 + \left(R_{3,0} + R_{3,1}\frac{d}{dt}\right) w_1 w_2 = 0 \tag{5.17}$$

Taking into account the product rule of differentiation, is is possible to rewrite (5.17) in the form

$$\begin{aligned} R_{1,0}w_1 + R_{1,1}\frac{d}{dt}w_1 + R_{2,0}w_2 + R_{2,1}\frac{d}{dt}w_2 \\ + R_{3,0}w_1w_2 + R_{3,1}w_1\frac{d}{dt}w_2 + R_{3,1}w_2\frac{d}{dt}w_1 = 0 \end{aligned} \tag{5.18}$$

which can then be formulated in the form of Definition 6 in w_1 as

$$\frac{d}{dt}w_1 = \frac{-\left(R_{1,0}w_1 + R_{2,0}w_2 + R_{2,1}\frac{d}{dt}w_2 + R_{3,0}w_1w_2 + R_{3,1}w_1\frac{d}{dt}w_2\right)}{R_{1,1} + R_{3,1}w_2} \tag{5.19}$$

under the condition that $w_2 \neq \frac{-R_{1,1}}{R_{3,1}}$.

The form of (5.19) allows for the application of the Picard-Lindelöf Theorem, Theorem 1, when considering the right hand side as $F(t, w_1)$. According to this theorem, for proof of existence and uniqueness, it suffices to show that $F(t, w_1)$ is continuous and satisfies the Lipschitz condition in w_1.

Recall that $F(t, w_1)$ is not defined in $w_2 = \frac{-R_{1,1}}{R_{3,1}}$. As the sum and quotient of continuous functions, $F(t, w_1)$ is continuous if w_2 is $\mathcal{C}^l$-continuous for a system of order l. Since $F(t, w_1)$ is linear in w_1 it is possible to find a constant $K \in \mathbb{R}$ such that (2.13) is fulfilled.

This shows that under the condition $w_2 \neq \frac{-R_{1,1}}{R_{3,1}}$ and assuming $\mathcal{C}^l$-continuity for w_2, a solution to the bilinear extended kernel representation exists and is unique. A similar argument applies to w_2. Further variables can be incorporated into the proof, each additional variable introduces one discontinuity to F for which the solution is not unique. In this way, for finite dimensional systems, a strong solution exists and is unique almost everywhere.[1]

[1]The existence of a weak solution is given in the sense of the existence of a series of strong solutions converging to the weak solution based on (Polderman and Willems, 1998, Th. 2.4.10). This indicates that the assumptions made in the proof do not limit the practical applicability.

The representation of bilinear systems in the Behavioural Framework developed above by making use of the extended bilinear kernel representation, to the best knowledge of the author, constitutes a novelty. While nonlinear systems are treated in a very theoretical manner in the literature, this well applicable class of nonlinear systems is not currently treated in the BF. Further the application to systems resembling practical systems closely, as presented in the sequel, is novel in both the model structure as well as the application itself.

5.3.4 Bond Graph representation

Having derived the differential and difference equations of the bilinear extended kernel representation, it is most natural to ask for a graphical representation of the system. This graphical representation can be expected to provide additional insight and will prove useful in analytical treatment of bilinear systems.

A block diagram representation of a bilinear system is presented in Figure 5.1. As discussed previously, the block diagram format is in general not suitable for application in an acausal setting, as it specifies the input/output structure of the system *a priori* to modelling.

Since in physical abstraction representations, a bilinear component may be included in atomic form, i.e. as a component defined by a behaviour including the bilinear term in its behavioural equations, the bond graph representation of a bilinear system is expected to yield more insight. Indeed, since among the components of bond graphs there is no element that multiplies bond variables as this would not be power conserving, the bond graph of a bilinear system has to contain an additional source of effort or flow to introduce the non-energy-conserving feature of a bilinear system.

This generation or absorption of energy does not become obvious in a block diagram setting, since the multiplication of the states, constituting generated or absorbed energy, is performed tacitly as the means are provided.

Assuming a bilinear first order system with continuous time axis $\mathbb{T} = \mathbb{R}$,

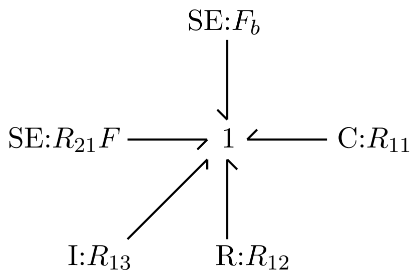


Figure 5.5: Continuous time bilinear system as a bond graph

a two dimensional signal space $\mathbb{W} = \{x, F\} \subseteq \mathbb{R}^2$ and the behaviour

$$\mathbf{B} = \left\{(x, F) \in \mathbb{W}^{\mathbb{R}} | R_{11}x + R_{12}\frac{d}{dt}x + R_{13}\frac{d^2}{dt^2}x + R_{21}F + R_{31}\frac{d}{dt}(Fx) = 0\right\} \tag{5.20}$$

with parameters $R_{ij} \in \mathbb{R}$, the bond graph contains the following elements:

- A capacitor C with capacity R_{11}
- A resistor R with resistance R_{12}
- An inertia I with inertia R_{13}
- A source of effort SE applying the force $R_{21}F$
- A source of effort SE applying the bilinear force $F_b := R_{31}\frac{d}{dt}(Fx)$

This bond graph is is depicted in Figure 5.5.

The behaviour of a system as defined above can be simulated utilising e.g. MapleSim. Since the provided components of this and similar software products is also based on energy conservation considerations, such systems cannot be assembled of the blocks in an acausal way.

For this reason, it is necessary to implement a bilinear system in e.g. the Modelica language, which is documented in detail in Appendix A.

The simulation result as shown in Figure 5.6 was achieved using the Modelica implementation, excited by a sinusoidal force of increasing amplitude. The asymmetric response to the symmetric excitation can be well observed on this figure as well as the state dependent behaviour, which are characteristics of bilinear systems.

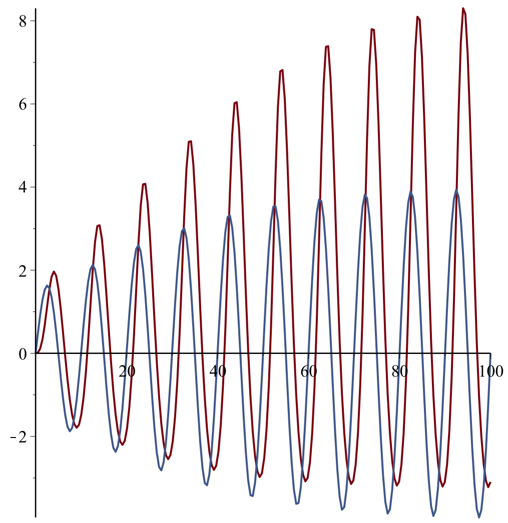


Figure 5.6: Simulation result of a system specified in a bilinear extended kernel representation

5.4 Application studies

5.4.1 Preliminaries

In order to put the novel model structure to a test in terms of applicability to practical systems, two application studies are conducted. These studies show an increasing degree of nonlinearity in the system to be modelled. The former is chosen due to the availability of recorded data, while the latter is selected since CSTR models form a popular benchmark for nonlinear modelling performance.

Both studies show that the bilinear extended kernel representation is able to model some practical systems appropriately and especially without determination of the input/output structure to be carried out before performing the modelling process.

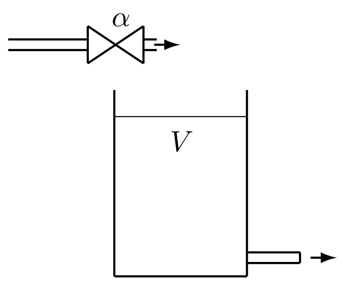


Figure 5.7: Gravity drained reservoir assembly

5.4.2 Application to a hydraulic system

Introduction

The system under consideration in this application study is a gravity drained reservoir, which is very similar to the single reservoir presented in Example 2. The difference is that the system has a fixed diameter drain pipe and has a nonconstant influx of the fluid via a control valve. This application was chosen due to its similarity with the interconnection paradox 2, while at the same time exhibiting mildly nonlinear behaviour.

Data recorded from such a system with parameters not specified is provided by Hedengren (2003), three disctinct data sets are provided. Analytically, this system can be modelled as $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ with time axis $\mathbb{T} = \mathbb{R}$, a signal space consisting of the volume $V \in \mathbb{R}^+$ and the valve position $\alpha \in [0, 100]$, i.e. $\mathbb{W} = \mathbb{R}^+ \times [0, 100]$ and the behaviour

$$\mathcal{B} = \left\{ (V, \alpha)^T \in \mathbb{W}^{\mathbb{T}} | \frac{d}{dt} V + R_{12}\sqrt{V} = R_{21}\alpha \right\} \tag{5.21}$$

Modelling

The nonlinear model structure above is assumed to be unknown, thus the system is modelled using linear and bilinear extended kernel representation of identical lag, in order to allow for a comparison. The identification is performed by help of a total least squares estimator, since a further assumption is that both variables are potentially corrupted by noise. In order to be able

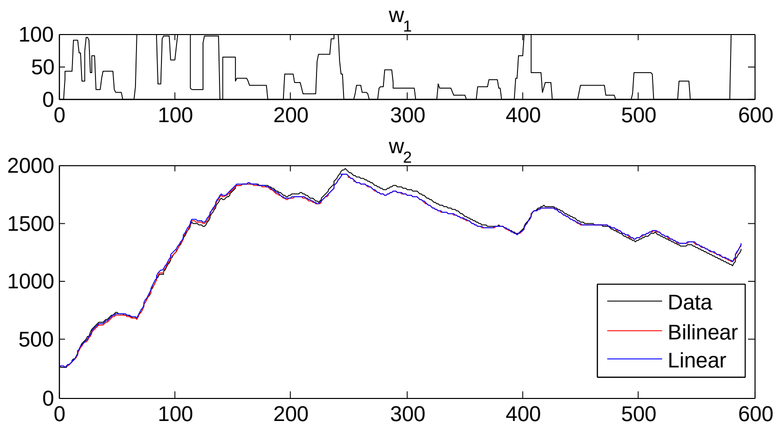


Figure 5.8: Model performance on the calibration data set

to approximate the nonlinear function in (5.21), a lag of 2 is chosen using empirical methods.

For identification purpose, one data set is set aside for model calibration, while another unseen set is used for validation. The calibration data set is shown in Figure 5.8, with both models simulated for comparison.

A linear kernel representation governed by the polynomial matrix

$$\begin{bmatrix} 1.00 - 1.14\sigma + 0.141\sigma^2 \\ 0.0016 + 0.0114\sigma - 0.228\sigma^2 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \tag{5.22}$$

is estimated as a linear model, while a bilinear extended kernel representation is governed by

$$\begin{bmatrix} 1.00 - 1.13\sigma + 0.131\sigma^2 \\ 0.0339 + 0.0221\sigma + 0.2657\sigma^2 \\ 10^{-4}(-0.2634 - 0.0782\sigma + 0.2823\sigma^2) \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \tag{5.23}$$

for an extended signal space $\mathbb{W}_{ext} = \mathbb{R}^+ \times [0, 100] \times \mathbb{R}^+$ and $w_3 := w_1 w_2$.

The performance on the validation data set is shown in Figure 5.9. In both figures, no significant difference becomes obvious, instead it can be found that both models are able to model the data appropriately. In the

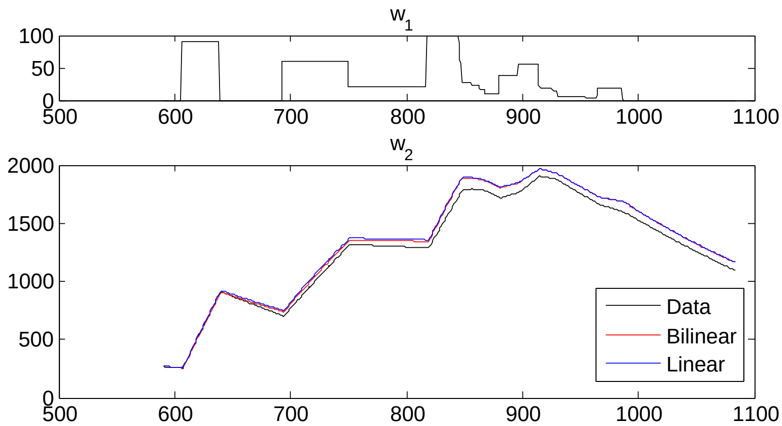


Figure 5.9: Model performance on the validation data set

corresponding error plot in Figure 5.10, the slightly better performance of the bilinear extended kernel representation is shown.

Indeed, when inspecting the errors between model and data, it is found that with a root mean square error of 1315 compared to 1441 for bilinear and linear model, respectively, an improvement of 9.5% is achieved. The result however is likely to be not significant enough to justify the use of nonlinear techniques in this particular application.

5.4.3 Application to a chemical system

Preliminaries

In order to further evaluate the applicability of the bilinear extended kernel representation, data originating from a nonlinear model of a continuous stirred tank reactor (CSTR) is modelled as a bilinear behavioural system.

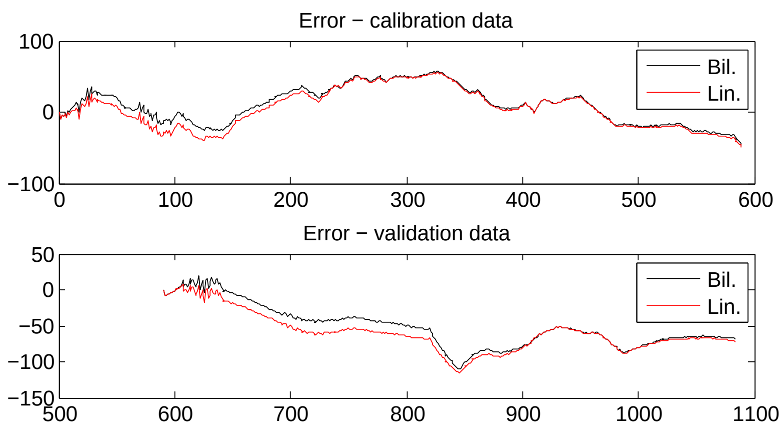


Figure 5.10: Error between recorded data and model results

The CSTR governed by (Henson and Seborg (1997))

$$\begin{aligned} \dot{C}_a &= \frac{q}{V}\left(C_{af} - C_a\right) - K_0 \exp\left(\frac{-\left(\frac{E}{R}\right)}{T}\right) C_a \\ \dot{T} &= \frac{q}{V}\left(T_f - T\right) - \\ &\frac{m}{\rho C_p} K_0 \exp\left(\frac{-\left(\frac{E}{R}\right)}{T}\right) C_a + \frac{UA}{V\rho C_p \left(T_c - T\right)} \end{aligned} \tag{5.24}$$

is simulated over $N = 1000$ steps with the coolant temperature T_c uniformly distributed from 235 K to 285 K. This variable is then normalized and in the following denoted as w_1. The concentration of substance A, C_a is already in the range $0 \ldots 1$ and is denoted w_2. Both variables recorded from the simulation are corrupted with normally distributed white noise with a variance of 0.02. The data is split into two data sets, both consisting of $N = 500$ data points.

Since the reaction A to B is an exothermic reaction in this model, it is not obvious to distinguish between input and output in this system, since an increase in temperature causes an introduction of more exothermic energy in the reactor, rising the temperature and decreasing the concentration of A in the reactor. At the same time, as long as enough substance B is present in the CSTR, an increase in A results in a temperature rise due to more reactions taking place, increasing the coolant temperature.

Depending on the operating range and type of reaction, chemical reactions in a CSTR can show highly nonlinear behaviour, including output multiplicities. The CSTR model will be discussed in more depth in Chapter 6. For the present study a parameter set is chosen that exhibits only invertible nonlinear behaviours.

Modelling

In the modelling and parameter estimation stage, both a linear and a bilinear extended kernel representation are estimated by the total least squares estimator. In order to have the same number of free parameters, a lag 1 bilinear extended kernel representation is compared to a lag 2 linear kernel representation.

The lag 2 linear kernel representation is estimated based on the calibration data set and given by

$$\begin{bmatrix} 1.00 - 1.86\sigma + 0.874\sigma^2 \\ 0.0160 + 0.0092\sigma - 0.0418\sigma^2 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \tag{5.25}$$

The lag 1 bilinear extended kernel representation is estimated from the calibration data set to

$$\begin{bmatrix} 1.00 + 1.49\sigma \\ -5.70 + 2.75\sigma \\ 5.93 - 2.43\sigma \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \tag{5.26}$$

The simulation results compared to the calibration data set for both models is shown in Figure 5.11. Here a more appropriate behaviour in the operational extremes can be observed, whereas in the more linear parts of the operating range both models perform well.

To evaluate the performance on unseen data, both the linear and the bilinear model are simulated over $N = 500$ time steps using the second part of the data from the simulation of the model (5.24) for comparison. The validation data set shown in Figure 5.12 shows that also here a significant improvement may be achieved by extending the signal space to comprise nonlinear variables.

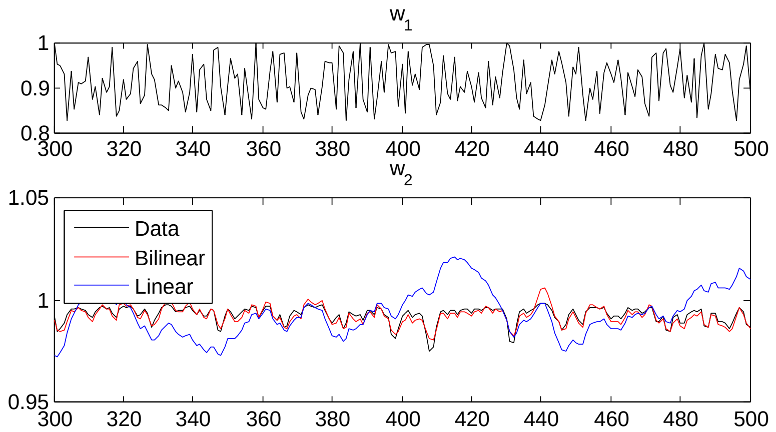


Figure 5.11: Portion of the calibration data set

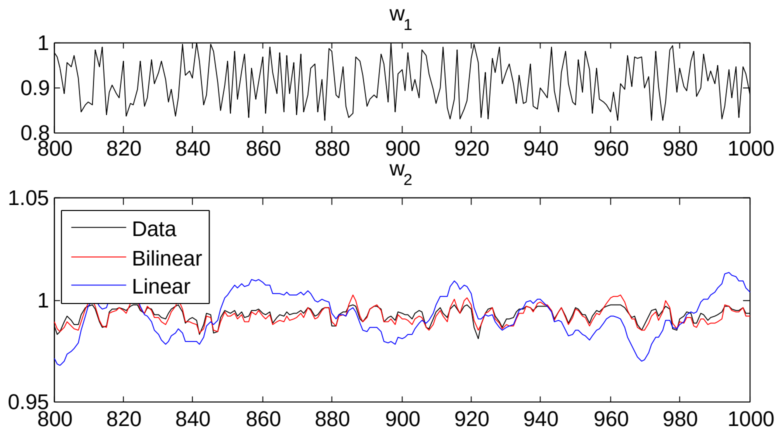


Figure 5.12: Portion of the validation data set

5.5 Concluding remarks

The validity of models largely depends on the features they are able to explain and generate. The dominant features of single model classes is investigated and extended to encompass a feature which is important for application in the BF. This feature is the existence of a unique inverse of the nonlinear model structure, that defines whether the model structure determines the input/output structure implicitly. It is found that bilinear models guarantee the existence of the inverse of the model equations.

While in the current literature in the BF, nonlinear systems are not excluded, no treatment of bilinear models is known to the author. This marks the introduction of the bilinear extended kernel representation as a novelty. In addition to this novel representation, the applications of the same to practical application examples are new to the field.

The existence and uniqueness of solutions of the respective difference or differential equation is investigated, the former by making use of solving the difference equations explicitly, the latter by application of the Picard-Lindelöf theorem. It is found that in both cases, the solution to the acausal formulations exists almost everywhere. Further insight is provided by inspection of a bond graph representation, here clearly the introduction or absorption of energy becomes obvious by the necessity of adding an additional source component to express the bilinear term. Both the proofs of existence and the representation of bilinear systems as a bond graph constitute a novelty.

The qualitative behaviour of the formulation is inspected by making use of a simulation of a bilinear extended kernel representation in MapleSim, based on a specifically developed bilinear component. Initial modelling studies on two data sets of varying nonlinearity are performed that indicate the applicability.

In this way, a promising first step towards the extension of model validity and applicability is taken by encompassing bilinear systems in an acausal formulation. This improved model validity is anticipated to promote the application of the BF to practical systems. To further extend this initial progress, investigation into the applicability of the novel structure for model based control purposes, similar to the approach chosen in (Martineau *et al.*, 2004), is required.

Chapter 6

A practical approach to approximate modelling

... very often the laws derived by physicists from a large number of observations are not rigorous, but approximate.

Augustin Louis Cauchy

6.1 Introduction

As a next step after the extension of the model classes available in the BF to encompass bilinear models in a manner directly applicable to practical control problems, the approximate modelling or system identification procedure is reviewed. For this purpose, an introduction to approximate modelling both in an input/output as well as in an acausal setting is provided. In the former, both the equation error and the errors-in-variables (EIV) paradigm is presented.

System Identification in the BF dates back to the foundational publications on the subject (Willems, 1986a,b, 1987). In these publications, a break in input/output thinking as well as a strict methodology was intended by the author, for which purpose a totally different approach, accompanied by a specific terminology was chosen. Following the approach of addressing the exact modelling case before turning to approximate models, Section 6.2.3 also introduces this terminology.

While the exact modelling approach by design is not fit for application to practical problems, since it relies on uncorrupted data and increases model complexity without limits, the approximate methodology in theory is more applicable. When analysing the procedures however, it turns out that the approach to either limit misfit or complexity may yield highly sensitive models. To overcome this drawback, already pointed out by (Willems, 1987, p. 89), a viable solution, also proposed in general form by Willems (ibid.), is developed. The combined misfit/complexity approach optimises the improvement in terms of misfit when increasing the model complexity and in this way terminates before unnecessary high sensitivities are reached. This approach resembles a manual approach, however can run automatically.

The novel approach is tested in a Monte Carlos simulation in conjunction with the bilinear extended kernel representation developed in the previous chapter. A comparison to a linear structure identified by the same method as well as an equation error structure is performed.

The application to the model of a nonlinear CSTR forms a step towards the application of the BF to practical systems.

6.2 Modelling

6.2.1 Introduction

The modelling process starts with the selection of the model class or model classes that serve as candidates for the later final model. As discussed previously, frequently linear causal models are selected despite their inability to express certain features of a system. In this sense, the extensions provided in the two previous chapters to this narrow set of models, together with the criteria for model selection, may serve to find more and richer model classes.

Once a class of candidate models is chosen, the most suitable model from this class of candidate models must be selected. This is typically carried out by determining the model parameters based on system data. This task is considered as an inverse problem (Aster *et al.*, 2012) as opposed to the forward problem of generating a model data set from an existing model.

In his well-accepted book (Ljung, 1999), Ljung defines system identification as a process envolving three entities:

1. Data: Data recorded from operation of the system to be identified, possibly stemming from a specifically designed experiment, possibly from normal operation of the system.

2. Candidate Models: The selection of a set of candidate models is considered the most crucial step, where engineering insight and *a priori* knowledge may be applied to improve the result of this modelling step. These candidate models may have different structures, but may also vary only in their parameters.

3. Determination of the best model in the set of candidate models: Based on the data, the model that explains the data best, in the sense of an appropriate metric, is chosen.

This approach is applicable for both causal and Behavioural Framework, however some differences exist in the selection of candidate models and the criteria for the model fit, determining the best model in the class under consideration.

6.2.2 Causal Framework

Equation error paradigm

In (Gevers, 2006) the time before 1960 is considered as early history of system identification. While the identification of the path of Ceres by C.F. Gauss (Gauss, 1809) can be considered as a truly early example exhibiting all characteristics mentioned above, the modern approach to system identification of dynamical systems within the area of automatic control is considered to begin in 1960. The basis for this development was the novel state space representation of systems, originating in the work of Kalman (Ljung, 1996). Before these formulations existed, mainly Bode-Nicols synthesis and related techniques were applied, a need for system identification techniques was not observed due to the lack of parametric representations.

Taking into account that the bases of the modern approach to the identification of dynamical systems were formulated before the advent of the BF and the *a priori* causal nature of the model representation that fostered its development, it appears natural that the underlying paradigm is causal. Deistler (Deistler, 2007) defines a *mainstream theory*, characterised by:

1. 'The model class consists of linear, time-invariant, finite dimensional, causal and stable systems only. The classification of the variables into input and output is given *a priori*.

2. Uncertainty is modelled by the use of stochastic models for noise. [...]

3. The observed inputs are free of noise and uncorrelated with the noise process.

4. The approach to estimation is semi-nonparametric in the following sense: In general the parameter space for describing system- and noise parameters will be not finite dimensional, since e.g. systems of arbitrarely [sic] high orders are considered. [...]

5. For the statistical analysis, emphasis is laid on asymptotic properties (consistency, asymptotic normality and asymptotic efficiency), mainly because finite sample properties are hard to obtain analytically.'

This mainstream theory is founded on the popular autoregressive model with exogeneous input (ARX) (Ljung, 1999):

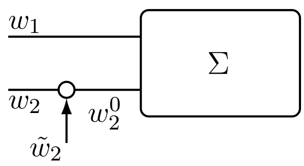


Figure 6.1: ARX system

Definition 17 (ARX model) *Be $\mathbb{T} = \mathbb{Z}$ a discrete time axis,*

$$\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_2^0 \times \tilde{\mathbb{W}}_2 \subseteq \mathbb{R}^4 \tag{6.1}$$

a signal space and

$$\begin{aligned}\mathcal{B} = \Big\{ \big(w_1, w_2, w_2^0, \tilde{w}_2\big) : \mathbb{T} \to \mathbb{W} \big| w_1(t) + a_1 w_1(t-1) + \cdots + a_{n_a} w_1(t - n_a) \\ = b_1 w_2^0(t-1) + \cdots + b_{n_b} w_2^0(t - n_b) \wedge w_2(t) = w_2^0(t) + \tilde{w}_2(t) \Big\}\end{aligned} \tag{6.2}$$

a full behaviour defined by linear difference equations. Then a system

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$$

is termed ARX system.

In (6.2) w_1 denotes the variable which is part of the autoregression while w_2 denotes the exogeneous variables. The error term $\tilde{w}_2$ represents the so-called equation error term, while w_2^0 is the noise-free exogeneous variable. Both w_1 and w_2 are manifest variables, while w_2^0 and $\tilde{w}_2$ are latent variables which are assumed to be unavailable for the system identification task. Figure 6.1 depicts an ARX system.

It becomes obvious from a close inspection of (6.2) that the form of an ARX model assumes a causality with w_1 causing w_2. For this reason, behaviours as defined in (6.2) are typically not used in the BF. Due to the random nature of $\tilde{w}_2$ and the fact that w_2^0 is part of the autoregression, both latent variables cannot be easily eliminated from (6.2).

The solution to the problem posed in the mainstream theory can be solved by linear regression by making use of the the parameter vector

$$\theta = [a_1 \cdots a_{n_a} \; b_1 \cdots b_{n_b}]^T \tag{6.3}$$

and the regression vector

$$\phi(t) = [-w_1(t-1) \cdots - w_1(t-n_a) \; w_2(t-1) \cdots w_2(t-n_b)]^T \tag{6.4}$$

With these vectors, it is possible to transform the linear difference equation in (6.2) to the predictor

$$\hat{y}(t|\theta) = \phi(t)^T \theta \tag{6.5}$$

for an estimated parameter vector θ.

The parameter vector θ can be estimated by minimising the prediction error

$$\epsilon(t,\theta) = w_2(t) - \phi(t)^T \theta \tag{6.6}$$

according to the least squares criterion

$$V_N(\theta) = \frac{1}{2N} \| \left(w_2(t) - \phi(t)^T \theta \right\|_2 = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} \left(w_2(t) - \phi(t)^T \theta \right)^2 \tag{6.7}$$

The 2-norm in (6.7) yields the advantage of making the minimisation possible analytically, the resulting least squares estimator based on N data points $\hat{\theta}_N^{LSE}$ is

$$\hat{\theta}_N^{LSE} = \arg\min V_N(\theta) = \left(\frac{1}{N} \sum_{t=1}^{N} \phi(t)\phi(t)^T \right)^{-1} \frac{1}{N} \sum_{t=1}^{N} \phi(t) w_2(t) \tag{6.8}$$

provided the inverse exists.

The family of ARX models and the equation error paradigm is commonly applied, due to its simplicity and applicability to e.g. model based control tasks. Since some of the underlying assumptions are not compatible with the BF, it is of limited use in this framework.

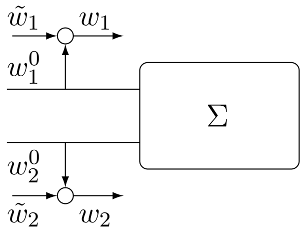


Figure 6.2: Errors-In-Variables Problem

Errors-In-Variables paradigm

The Errors-In-Variables (EIV) Paradigm lays its foundation on a violation of axiom 3) in the mainstream theory above, i.e. it assumes that both input and output are corrupted by noise. This change in paradigm can be motivated by the following reasons (Söderström, 2007)

- A model of the system dynamics between noise-free input and noise-free output is desired.
- The understanding of the dynamical system is valued higher than a good prediction.
- The requirement to analyse the data as if the experiment was not performed by the person running the identification.
- The user lacks enough information to classify the signals into input and output.

The relation between the last reason and the BF is evident, while the remainder does not necessarily stem from a consideration of the system following the BF.

The EIV paradigm differs from the Equation Error paradigm in that it assumes the errors to be added to all variables of a system, without regard to whether these are considered as input or output. Such a system is represented graphically in Figure 6.2, here the noise free variables are marked with the superscript ^{0} while the noise sequences bear a tilde.

This graphical problem formulation is frequently formalised to yield the following definition

Definition 18 (EIV ARX model) *Let $\mathbb{T} = \mathbb{Z}$ be a discrete time axis,*

$$\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_1^0 \times \tilde{\mathbb{W}}_1 \times \mathbb{W}_2 \times \mathbb{W}_2^0 \times \tilde{\mathbb{W}}_2 \subseteq \mathbb{R}^6 \tag{6.9}$$

a signal space and

$$\begin{aligned} \mathcal{B} = \Big\{ \big(w_1, w_1^0, \tilde{w}_1, w_2, w_2^0, \tilde{w}_2\big) : \mathbb{T} \to \mathbb{W} \big| \\ w_1^0(t) + a_1 w_1^0(t-1) + \cdots + a_{n_a} w_1^0(t-n_a) \\ = b_1 w_2^0(t-1) + \cdots + b_{n_b} w_2^0(t-n_b) \wedge w_1 = w_1^0 + \tilde{w}_1(t) \wedge w_2 = w_2^0 + \tilde{w}_2(t) \Big\} \end{aligned} \tag{6.10}$$

a behaviour defined by linear difference equations. Then a system

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$$

is termed EIV ARX system. In this context, $\tilde{w}_i$, $i = 1, 2$ denotes the error terms added to both variables, these are considered latent variables as are w_i^0, $i = 1, 2$, the noise-free variables.

The system identification task for the EIV ARX system differs from that of the Equation Error paradigm in that it does not suffice to minimise the prediction error on the output. While the canonical parameter estimator for the equation error paradigm, the linear least squares estimator, solves the minimisation problem (Söderström, 2007)

$$\min \|\triangle b\|^2 \quad \text{s.t.} \quad Ax_{LS} = b + \triangle b \tag{6.11}$$

The EIV paradigm canonically applies a Total Least Squares (TLS) estimator. The TLS estimates a parameter vector x_{TLS} by minising the problem

$$\min \|\triangle A \triangle b\|_F^2 \quad \text{s.t.} \quad (A + \triangle A)x_{TLS} = b + \triangle b \tag{6.12}$$

with $\|\cdot\|_F$ denoting the Frobenius norm, which for the vector $\triangle b$ is identical to the Euclidean norm.

In a 2-dimensional space with a regression problem, this relates to the difference between the error parallel to one axis as opposed to minimising the error along both axes, orthogonal to the line that minimises the error. This is illustrated in Figure 6.3 and can be extended to n dimensions and

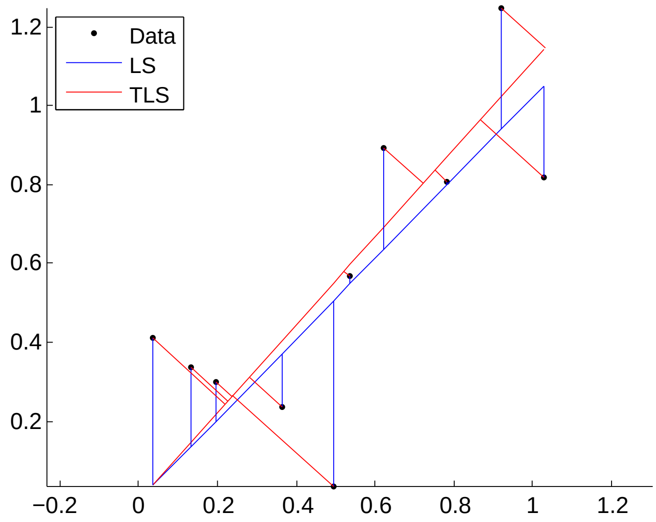


Figure 6.3: Comparison of Linear (LS) and Total Least Squares (TLS) Error Metrics

the problem of fitting a hyperplane. The type of error measure, which is maintained orthogonal to the hyperplane to be identified, leads to the term *orthogonal regression.*

To identifiy the EIV ARX system as in Definition 18, an extended parameter vector

$$\theta = [1\, a_1 \,\cdots\, a_{n_a}\; b_1 \,\cdots\, b_{n_b}]^T \tag{6.13}$$

and an extended regression vector

$$\phi(t) = [-w_1(t-1) \,\cdots\, -w_1(t-n_a)\; w_2(t) \,\cdots\, w_2(t-n_b+1)]^T \tag{6.14}$$

is employed to be able to express (6.10) as a linear regression

$$w_1(t) = \phi(t)^T\theta + \epsilon(t) \tag{6.15}$$

The error term $\epsilon(t)$, instead of being added only to the equation, collects the errors in both variables:

$$\begin{aligned}\epsilon(t) = [&\tilde{w}_1(t)\; a_1\tilde{w}_1(t-1) \;\cdots\; a_{n_a}\tilde{w}_1(t-n_a) \\ &b_1\tilde{w}_2(t) \;\cdots\; b_{n_b}\tilde{w}_2(t-n_b+1)]^T\end{aligned} \tag{6.16}$$

Since these vectors were constructed to collect the terms of the difference equation in (6.10) on the left hand side (consequently expressing it in kernel form), it is possible to transform the linear difference equation into the linear regression

$$\phi(t)^T\theta \approx \phi^0(t)^T\theta^0 = 0 \tag{6.17}$$

assuming exactly known parameter and regression vectors θ^0 and ϕ^0.

The parameter vector θ_{TLS} can be found by making use of the TLS method estimating

$$\Phi_N\theta \approx 0 \tag{6.18}$$

with an observation matrix

$$\Phi = \left(\phi(1)^T \cdots \phi(N)^T\right)^T \tag{6.19}$$

While in (Söderström, 2007) the inappropriateness of the TLS estimator for dynamical systems due the structural similarity of the lines of Φ is stressed, it is used here to illustrate the EIV principle. Beside this potential drawback depending on the system under study, the error and the excitation signal, the TLS method offers a symmetric treatment and yields valuable algebraic information, e.g. the singular values. Therefore the TLS method is considered as the canonical estimator under the EIV paradigm.

6.2.3 Modelling in the Behavioural Framework

Introduction

The BF employs a different concept of modelling than the input/output framework. Instead of considering the data as subject to randomized errors, the misfit between measured data and model data is assumed to be also caused by choosing a model class not containing the true system representation (Willems, 1986b). This concept of misfit leads to a different approach to model selection. The observed data is modified to allow for selection of a model from the set of candidate models, whereas in the input/output framework, the model is changed to explain the data.

This view on misfit caused by measurement noise and model mismatch is highly relevant for adaptive control, where deliberately a simpler model than the system representation is chosen and adapted over time.

Modelling in the BF is treated first in (Willems, 1986b) and (Willems, 1987), both articles being limited to linear time-invariant models. Interestingly, Willems addresses the approximate modelling issue by working on the theory of exact modelling first. In recent approaches, approximate modelling in the BF is considered mainly within the EIV framework (Söderström, 2007).

With respect to the acausal nature of the BF, the EIV framework is well suited for identification in the BF since due to the acausality no input/output structure can be determined. While the BF due to its model structures needs to rely on errors-in-variables identification, the basic assumptions of an *a priori* symmetric treatment of the variables in the underlying model is not *ab initio* applied in EIV literature, see (Söderström, 2007) and (Linden *et al.*, 2006). Further, the EIV framework does not apply the general concept of misfit as proposed by Willems. However, many of the techniques are well adapted to the BF, its acausal nature and its model structures.

Terminology and exact modelling

In order to set out without an *a priori* concept about the causal structure of a system to be modelled, the language is revised to avoid any causality being assumed beforehand and follows (Willems, 1986b). In this article, the system identification methodology is approached in an exact modelling approach, which is later generalised to the approximate modelling approach. This is opposed to the history of formalised modelling, where the inverse problem was solved approximately almost since the beginning.

Assuming a *phenomenon* is to be modelled from e.g. recorded data, this phenomenon needs to be quantified initially. This is carried out by making use of a phenomenon set S, the elements of which are termed *attributes*, i.e. all possible data. Considering a resistor as an example, an appropriate choice would be $S = \mathbb{R}^2$, the set of all 2-tuples of voltage and current. For dynamical systems defined by $\Sigma = \{\mathbb{T}, \mathbb{W}, \mathcal{B}\}$, the set of attributes is $S = \mathbb{W}^{\mathbb{T}}$. While (Willems, 1986b) uses a phenomenon set, frequently S is a real vector space, in which case it will be termed *phenomenon space* and a set of basis vectors will be termed *components* of this phenomenon.

Narrowing down the set of attributes declared possible by the system to be modelled, one searches for a *model* $M \subset S$. This usually is supposed to be stemming from a *class of models* $\mathcal{M} \subset 2^S$. As an example, consider the task of identifying a resistor. One would typically record current-voltage pairs and, based on some knowledge about Ohm's law, would choose

$$\mathcal{M} = \big\{(U, I) \in \mathbb{R}^2 | U = RI,\, R \in \mathbb{R}\big\}$$

This choice restricts the set of models from all sets of 2-tuples (constituting 2^S) to a set of linear subspaces.

A *measurement* is an observation of an attribute of a phenomenon, forming a set Z. By definition, they are assumed to be noise-free. Due to their definition as noise free, $Z \subset S$ holds. It is not necessary that the measurements are experimental data, it is also possible to consider data summarising observations. It is thus possible to consider the TLS estimate of a linear subspace a measurement in the sense that it is the best possible estimate of a noise free measurement. The measurements Z form a class of sets $\mathcal{Z} \subset 2^S$. This class of measurements may be restricted to reflect any limitations, e.g. boundedness or dynamic constraints.

In an exact modelling context, measurements Z can falsify or *unfalsify* a model M. In the latter case, $Z \subset M$ holds.

In Figure 6.4, this vocabulary is visualised for the two dimensional case, which can be imagined as the task of modelling a resistor from measured voltage-current pairs. The attribute axes span the phenomenon space $S = \mathbb{R}^2$. In this space, two model classes are depicted by some of their members, $\mathcal{M}_1 = \{(U, I)|U = RI\}$ and $\mathcal{M}_2 = \{(U, I)|U = c\}$. The models M_i, $i = 1, 2, 3$ belong to $\mathcal{M}_1$ and $M_4 \in \mathcal{M}_2$. The measurements Z falsify the models M_i, $i = 2, 3, 4$, while M_1 explains the measurements.

With the set theoretic view of a model M, it is always possible to generate a model M' that contains more attributes, i.e. $M' \setminus M \neq \emptyset$. It is common to consider a model more powerful the more restrictive it is. Having informally introduced the order relation 'more powerful than' for model classes, a natural question is for the maximum of the set $\mathcal{M}$, leading to the following definition.

Definition 19 (Most powerful unfalsified model) *Be Z a set with measurements of a phenomenon and $\mathcal{M}$ a set of models. A model $M_Z^* \in \mathcal{M}$ is termed most powerful unfalsified model (MPUM) in $\mathcal{M}$ for the measurements Z if $Z \subset M_Z^*$ and*

$$Z \subset M \in \mathcal{M} \;\Rightarrow\; M_Z^* \subset M$$

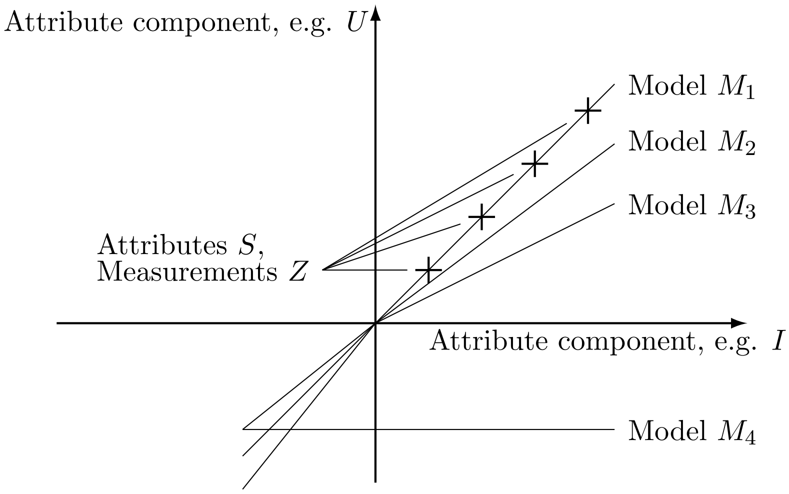


Figure 6.4: System Identification vocabulary in the BF

The existence of the MPUM in a model class $\mathcal{M}$ is given if for all model subclasses $\mathcal{M}' \subset \mathcal{M}$ the intersection of the contained models lies with $\mathcal{M}$, i.e.

$$\mathcal{M}' \subset \mathcal{M} \;\Rightarrow\; \left(\bigcap_{M \in \mathcal{M}'} \in \mathcal{M} \right)$$

and $\mathcal{M}$ contains at least one unfalsified model M for a set of measurements Z

$$\forall Z \in \mathcal{Z}\; \exists\, M \in \mathcal{M} :\; Z \subset M$$

This existence theorem is subject of (Willems, 1986b, Prop. 11).

Approximate modelling

The definitions introduced above were introduced for the exact modelling case, which is, on its own, not of too much relevance for practical purposes. The general concept and the associated terminology however transfer well to the approximate modelling task. Only the notion of a model being unfalsified by measurements and consequently that of the MPUM cannot be applied to the approximate modelling case, since the assumed misfit between measurement and model does not admit falsification based on measurements only.

The order relation 'more powerful than' on the set of unfalsified models serves to select the one that is the best model in the model class according to the scientific approach of having very restrictive models. A model with misfit cannot be restrictive in the binary sense of being a set member or not, it rather assigns different degrees of relevance to the individual measurements and thus may not even be unfalsified by a single measurement it is based on. This calls for another way of ordering the set of models and thus selecting the most appropriate model in class.

In the case of approximate modelling, it is commonly assumed that the predictive power of the model decreases as the number of free parameters is increased. This is reflected in the following definition (Willems, 1987).

Definition 20 (Complexity mapping) *For a phenomenon S, a model class $\mathcal{M} \subset 2^S$ and $\mathcal{Z} \subset 2^S$ a class of measurements, the complexity c is a mapping*

$$c : \mathcal{M} \mapsto \mathcal{C}$$

from the model class to the the complexity level space $\mathcal{C}$.

Not only the complexity plays a role in model selection, also the degree of fit, leading to a misfit mapping defined below.

Definition 21 (Misfit mapping) *With S, $\mathcal{M}$ and $\mathcal{Z}$ as in Definition 20, define the misfit ϵ as a mapping*

$$\epsilon : \mathcal{Z} \times \mathcal{M} \mapsto \mathcal{E}$$

with $\mathcal{E}$ the misfit level space.

Both the misfit and the complexity level space $\mathcal{E}$ and $\mathcal{C}$ are partially ordered spaces. The most general idea of both mappings is provided by applying the ordering induced by inclusion on the set $\mathcal{Z}$. For $M_1, M_2 \in \mathcal{M}$ this yields

$$M_1 \subset M_2 \ \Rightarrow c\,(M_1) \leq c\,(M_1) \tag{6.20}$$

$$M_1 \subset M_2 \ \Rightarrow \epsilon\,(M_1) \leq \epsilon\,(M_1)\,,\ \forall Z \in \mathcal{Z} \tag{6.21}$$

Both high complexity, leading to a model with small explanatory power, and high misfit are undesired in modelling and in fact the human modeller tends to weigh one against the other. A similar concept, that of a *utility function* is proposed in (Willems, 1987, p. 89) but is not followed as it is considered difficult to find an intuitively justifiable utility function.

In (Willems, 1987), two fundamental algortithms are proposed, each optimising the criteria of Definitions 20 and 21 separately while keeping the other criterion below a defined threshold. Consequently, the algorithms are termed *modelling with limited complexity* and *modelling with limited misfit.* The former limits the allowed complexity of the models, while the latter fits a model up to a given misfit.

For the following exposition of the modelling methodologies, assume a modelling set-up consisting of the sets S, $\mathcal{M}$, $\mathcal{Z}$ and the mappings $\epsilon : \mathcal{Z} \times \mathcal{M} \mapsto \mathcal{E}$ and $c : \mathcal{M} \mapsto \mathcal{C}$.

Modelling with limited complexity

Assuming a fixed *maximal admissible complexity* c^{adm}, the *optimal approximate model* $M^* \in \mathcal{M}$ in the model class $\mathcal{M}$ based on measurements $Z \in \mathcal{Z}$ satisfies the following conditions

1. $c\left(M^*\right) \leq c^{adm}$
2. $\left\{M \in \mathcal{M}, c(M) \leq c^{adm}\right\} \Rightarrow \left\{\epsilon\left(Z, M^*\right) \leq \epsilon\left(Z, M\right)\right\}$
3. $\left\{M \in \mathcal{M}, c(M) \leq c^{adm}, \epsilon\left(Z, M^*\right) = \epsilon\left(Z, M\right)\right\}$
   $\Rightarrow \left\{c\left(M^*\right) \leq c(M)\right\}$

This relates to an optimal approximate model which satisfies the required maximum admissible complexity and within this class minimises the misfit between model and measurements. In the event of more than one model satisfying the above, the one with the minimum complexity is chosen.

The complexity in this context can defined as follows (Willems, 1987, Definition 5).

Definition 22 (Complexity) *Be* $\{\mathbb{T}, \mathbb{R}^q, \mathcal{B}\}$ *with* $\mathcal{B}$ *a linear shift invariant closed subset* $\left(\mathbb{R}^q\right)^{\mathbb{T}}$ *equipped with the topology of pointwise convergence and* $\mathcal{B}_t = \mathcal{B}\big|_{\mathbb{T}\cap[o,t]}$ *the restriction of* $\mathcal{B}$ *to the first* t *sample instances. The complexity is defined as*

$$c_t = \frac{\dim \mathcal{B}_t}{q(t+1)}$$

The identification of an (AR) relation fulfilling the above is described generically in (Willems, 1987, Algorithm 8). This algorithm relies on the Singular Value Decomposition (SVD) to calculate the singular values of the truncated correlation matrizes for different complex lag structures and the corresponding singular vectors.

The dominant input data to the algorithm is a time series $\tilde{\mathbf{w}}$ and a complexity series

$$\mathbf{c}^{adm} = \left(c_0^{adm}, c_1^{adm}, \ldots, c_t^{adm}, \ldots\right),\ 0 \leq c_t^{adm} \leq 1$$

This complexity series is transformed to the largest non-negative monotonically non-decreasing convex sequence $\tilde{\mathbf{d}}_t$ with $\tilde{\mathbf{d}} \leq q(t+1)\mathbf{c}^{adm}$ for optimisation among other auxiliary sequences.

During the recursive part of the algorithm, the (AR) representations for the different complexities are calculated by making use of the SVD as the left singular vectors. The orthogonal complement of the truncated behaviour and its relative increase in the recursive step are examined to obtain the increase in complexity in the current step.

The algorithm terminates if either the further increase of the lag structure of the assumed model does not yield additional information, i.e. the rank of the observation matrix does not increase under this operation, or the assumed maximum admissible complexity is reached. The algorithm returns a set of polynomials corresponding to the (AR) relations identified and the the stage of termination. This enables the selection of the one optimal model under a given maximum complexity.

The algorithm is proven to operate generically correct and terminates in a finite number of steps, while the optimal approximate (AR) relation is always defined.

Modelling with limited misfit

Alternatively, assume a model to be adjusted to measurements up to a maximum misfit ϵ^{tol}. A model $M^* \in \mathcal{M}$ is termed *optimal approximate model* in a model class $\mathcal{M}$ for $Z \in \mathcal{Z}$ if it satisfies the conditions below:

1. $\epsilon\left(Z, M^*\right) \leq \epsilon^{tol}$
2. $\left\{M \in \mathcal{M}, \epsilon(Z,M) \leq \epsilon^{tol}\right\} \Rightarrow \left\{c\left(Z, M^*\right) \leq c\left(Z, M\right)\right\}$

3. $\left\{M \in \mathcal{M}, \epsilon(Z, M) \leq \epsilon^{tol}, c\left(M^{*}\right)=c(M)\right\}$
$\Rightarrow\left\{\epsilon\left(Z, M^{*}\right) \leq \epsilon(Z, M)\right\}$

This can be interpreted such that the optimal approximate model has a tolerated error level. Within the class of models having this error level, it minimises the complexity. In case there exists more than one model exhibiting these properties, the one with smallest misfit is chosen.

An identification algorithm according to this principle is presented as (Willems, 1987, Algorithm 9). Similar to the algorithm described above, it relies on singular values of the truncated correlation matrix to assess the misfit between the time series and the behaviour at the current step of the algorithm. The classification of the truncated behaviours at the current step of the algorithm is also achieved by making use of the orthogonal complement.

Input data to the algorithm is a time series $\tilde{\mathbf{w}}$ and a misfit sequence

$$\epsilon^{adm}=\left(\epsilon_{0}^{adm}, \epsilon_{1}^{adm}, \ldots, \epsilon_{t}^{adm}, \ldots\right),\ \epsilon_{t}^{adm} \geq 0$$

During the recursive part of the algorithm, the square roots of the singular values of the truncated correlation matrix are compared to the tolerated msfit at the current step of the algorithm. All (AR) relations providing a misfit less than the tolerated ones are stored. The algorithm terminates when either the singular values indicate that an (AR) relation of misfit less than tolerated is found or the identified behaviour spans the whole observed behaviour.

The return values of the algorithm is a set of polynomials defining the (AR) relations at the steps of the algorithm and variables defining the selection of components of the algorithms, defining the optimal approximate (AR) relation.

The algorithm terminates after a maximum number of iterations, however the recursive substages may be executed arbitrarily often. Further, since the optimal approximate behaviour is defined at step 0, the algorithm always returns an optimal approximate model.

6.3 Approximate modelling algorithm with adaptations for improving applicability

6.3.1 Introduction

As pointed out earlier in this chapter and in (Willems, 1987), the modelling algorithms described above may terminate at a model that optimises the given target, i.e. either complexity or misfit while the other variable is kept constant, but is very sensitive or does not fit the data appropriately. In order to overcome this drawback, the conceptually proposed *utility function* approach is followed in this section to develop the combined misfit/complexity approach.

6.3.2 Modelling algorithm

While both algorithms described above form a sound theoretical basis and are working algorithms, the practising control engineer would rarely apply them without heuristic intervention. Typically, arbitrarily high model orders are undesired, even at the cost of some higher misfit. As the sensitivity of the singular vectors increases inversely proportional to the relative improvement, the algorithm should terminate when no large improvement in the singular values and thus in the misfit is obtained by increasing the order further.

The interesting aspect of the algorithms for modelling with limited misfit and limited complexity is the quantitative evaluation of degree of fit without simulation by inspection of the singular values of the observation matrix, denoted H, see Equation (6.22). This aspect is maintained while the sensitivity issues are addressed.

The amendment to the algorithm is the termination of the search via the improvement of fit of the model, i.e. the derivative of misfit with respect to complexity. This allows the algorithm to more closely resemble an engineering approach.

The input to the algorithm is an observed vector valued time series $\mathbf{w}$ and a minimum improvement $\delta_{min} > 0$. In the sequel, let $\cdot^0$ denote a noise free variable, while $\tilde{\cdot}$ denotes the respective noise sequence and time series without accent denote possibly noisy signals.

With the observation matrix

$$H_{t'}(w) = \begin{bmatrix} \mathbf{w}(0) & \mathbf{w}(1) & \cdots & \mathbf{w}(t') \\ \mathbf{w}(1) & \mathbf{w}(2) & \cdots & \mathbf{w}(t'+1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}(t) & \mathbf{w}(t+1) & \cdots & \mathbf{w}(t'+t) \end{bmatrix} \tag{6.22}$$

and a truncated observation matrix

$$\Pi_{t'} = \frac{1}{\sqrt{t}} H_{t'} H_{t'}^T, \tag{6.23}$$

define the rate of improvement $\delta(t')$ of the model as

$$\delta(t') = \frac{\epsilon\,(t'-1) - \epsilon\,(t')}{c\,(t') - c\,(t'-1)} \tag{6.24}$$

Here the misfit is defined as $\epsilon(t') = \sqrt{\lambda}$, where λ is the smallest nonzero singular value of $\Pi_{t'}$. As a difference from Definition 22, the complexity of the model is defined as

$$c(t) = \frac{\dim \mathcal{B}}{q}$$

During improvement of the algorithm, $\delta(t') > \delta_{min} > 0$ and the algorithm terminates once $\delta(t') < \delta_{min}$ or $\delta(t') < 0$. The last valid t' for which $\delta(t') > \delta_{min}$ defines the minimum order of a system improving the modelling performance while taking into account the rising complexity. The kernel representation can be calculated by use of the left singular vector associated with the smallest nonzero singular value. The algorithm can be outlined as in Algorithm 1.

As opposed to the two algorithms described in Willems (1987), this algorithm cannot guarantee a particular misfit, it rather terminates the search once the improvement becomes small or even negative. In this way, the algorithm can be configured such that it returns low-order, less sensitive models which may be used to address practical problems. No formal proof of convergence can be obtained. Based on the data and the selected model class, even diverging cases could be found. Further the algorithm does not attempt to obtain an optimum solution, it rather terminates at satisfying solutions.

An analysis and application of this combined misfit/complexity algo-

Data: Time series $\mathbf{w}$, minimum improvement δ_{min}
Result: Lag of optimal model t', polynomial matrix defining kernel representation R
Set $t' = 0$;
Calculate $\epsilon(0), \epsilon(1), \delta(0), \delta(1)$;
Calculate $\delta(1)$;
while $\delta(t'+1) > \delta_{min}$ **do**
  Calculate $\epsilon(t'+1), c(t'+1)$;
  Calculate $\delta(t'+1)$;
  Increase t' by 1;
end
Determine coefficients of polynomial matrix R for lag t' using TLS;

Algorithm 1: Combined misfit/complexity algorithm

rithm as defined above is presented in Section 6.4. While the original algorithms are explained in Willems (1987) without distinction between calibration and validation data, in this work the model is validated on unseen data. Furthermore, whereas only linear time invariant models are treated in the original paper, in this work two distinct model classes are compared according to the above procedure.

The development, testing and application of the present algorithm constitutes a novelty. Indeed, the application of a utility function instead of the isolated treatment of complexity and misfit was proposed in (Willems, 1987, p. 89),

> 'Of course, most appealing of all is to have a methodology in which a combination of complexity and misfit is used in a utility function $\mu : \mathcal{C} \times \mathcal{E} \rightarrow \mathcal{U}$ yielding $u(M, Z)$ to be maximized. However, this will not be pursued here since it seems difficult to come up with an intuitively justifiable utility function.'

The approach chosen in this thesis chooses a more intuitive approach in that, instead of maximizing some function, the modelling process is terminated as soon the improvement is marginal.

6.4 Numerical studies

6.4.1 Preliminaries

In order to show the applicability of the combined identification algorithm and the extended bilinear kernel representation, a simulation study is conducted. The example is an acausal sytem governed by

$$R_s(\sigma)\begin{pmatrix} w_1^0 \\ w_2^0 \end{pmatrix} = 0 \tag{6.25}$$

where

$$\begin{aligned} &R_s(\sigma) = \\ &\begin{pmatrix} -0.0885 - 0.1685\sigma - 0.0788\sigma^2 + 0.0385\sigma^3 \\ 1 - 1.2\sigma + 0.81\sigma^2 - 0.27\sigma^3 \end{pmatrix} \end{aligned} \tag{6.26}$$

The variable w_1^0 is assumed to be generated by an uniformly distributed random sequence $u = \mathcal{U}_{[0,1]}$ which is input to an ARX system governed by

$$w_1^0(k) = \frac{1.2}{2-\sigma}u(k) \tag{6.27}$$

and the behaviour of w_2^0 is simulated accordingly.

The variable w_2^0 is then distorted by a static nonlinearity to form a Wiener system, the static nonlinearity defined by

$$w_3^0(k) = \tanh\left(w_2(k)\right) \tag{6.28}$$

resembling a smooth saturation which may be uniquely inverted.

The noise free variables w_1^0 and w_3^0 are disturbed by mutually uncorrelated additive Gaussian distributed white noise with a standard deviation of 0.02, yielding the noisy manifest variables w_1 and w_3, respectively. These variables are considered as external signals, exclusively available for identification. This setup is depicted in Figure 6.5.

This Wiener system is simulated over $N = 600$ time steps, with the first 100 samples discarded to eliminate transient effects. This experiment is repeated for $M = 500$ times to allow for a Monte Carlo analysis of the modelling performance.

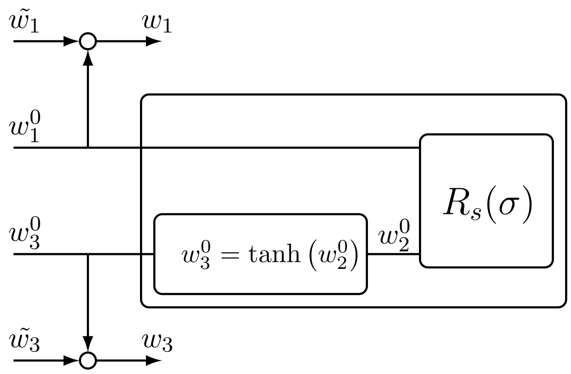


Figure 6.5: Diagram of the system for data generation, only the acausal parts of the model are considered as noise-free, noise is added to the manifest variables w_1 and w_3

6.4.2 Modelling

Algorithm 1 as described in Section 6.3.2 is applied to the data gained by the simulation, with half of the data left unseen by any part of the identification algorithm to form a validation data set. The first model class under consideration is the class of linear systems

$$\Sigma_{lin} = \left(\mathbb{T}, \mathbb{W}_{lin}, \mathcal{B}\left(R_{lin}\right)\right)$$

with the behaviour

$$\mathcal{B}\left(R_{lin}\right) = \left\{ \left(w_1, w_3\right)^T \in \mathbb{W}_{lin} \middle| R_{lin}(\sigma) \begin{pmatrix} w_1 \\ w_3 \end{pmatrix} = 0 \right\}, \tag{6.29}$$

a discrete time axis $\mathbb{T} = \mathbb{Z}$ and the signal space $\mathbb{W}_{lin} = \mathbb{R}^2$.

Another model class to be considered is that of bilinear systems defined in the bilinear extended kernel representation. These employ the same time axis as the linear systems and an extended signal space $\mathbb{W}_{bil} = \mathbb{R}^3$, yielding the bilinear system

$$\Sigma_{bil} = \left(\mathbb{T}, \mathbb{W}_{bil}, \mathcal{B}\left(R_{bil}\right)\right)$$

with the full behaviour

$$\mathcal{B}\left(R_{bil}\right) = \left\{ \left(w_1, w_3, w_4\right)^T \in \mathbb{W}_{lin} \middle| R_{bil}(\sigma) \begin{pmatrix} w_1 \\ w_3 \\ w_4 \end{pmatrix} = 0 \right\} \tag{6.30}$$

where the latent bilinear term is

$$w_4(k) = w_1(k)w_3(k) \tag{6.31}$$

Remark 1 (Complexity measure) *As in the present case, candidate models stemming from two distinct model classes are compared, it is necessary to extend the order relation 'more powerful than' to this case. Let $\mathcal{M}_l$ and $\mathcal{M}_b$ be the classes of linear and bilinear models, respectively. Assume a generic linear model of lag t denoted $M_{l,t} \in \mathcal{M}_l$ and a generic bilinear model $M_{b,t} \in \mathcal{M}_b$. It is easy to see that if the appropriate parameters are set to 0, each bilinear system contains the linear systems of identical lag, yielding the inclusion*

$$M_{l,t} \subset M_{b,t} \quad \forall t \mathbb{N}$$

On the other hand, a bilinear system of lag $t-1$ cannot express all features of a linear system of lag t, thus

$$M_{l,t} \not\subset M_{b,t-1}$$

For this reason, the order induced by inclusion on the united model class $\mathcal{M} = \mathcal{M}_l \cup \mathcal{M}_b$ is incomplete.

The same applies to the complexity measure $c(t) = \frac{\dim \mathcal{B}}{q}$ proposed for use in the combined misfit/complexity approach. Due to the fact that both model structures are linear in the parameters, for each new parameter (and the associated lagged variable), one more dimension is added to the behaviour. The following relations concerning the dimensions of the linear and bilinear models hold

$$\dim(\mathcal{M}_{l,t}) - \dim(\mathcal{M}_{l,t-1}) = 2q \tag{6.32}$$

$$\dim(\mathcal{M}_{b,t}) - \dim(\mathcal{M}_{b,t-1}) = 3q \tag{6.33}$$

For this reason, there exist models that have the same complexity, but are not identical, e.g. for $q = 2$, a lag 2 bilinear model has the same complexity as a lag 3 linear model.

While this remark is of little importance for the practical application of the algorithm presented here, bearing in mind especially that the aim is to compare the two model classes, it shall be noted that the extension of model classes to encompass nonlinear models leads to problems such as that of deciding, in the case of identical performance, which model structure is to be selected.

Both models stemming from the BF and identified according to the combined misfit/complexity algorithm shall not only be compared to each other, further a model structure, together with identification procedure, from an input/output setting is applied to model the data.

To this aim an equation error (EE) model is estimated following Ljung (1999). The structure of the EE model is chosen to be

$$w_3(t) = \frac{b_1 + b_2\sigma^{-1} + \cdots + b_{n_b}\sigma^{-n_b+1}}{1 + f_1\sigma^{-1} + \cdots + f_{n_f}\sigma^{-n_f}} w_1(t) + e(t) \tag{6.34}$$

where $e(t)$ denotes the error. The orders of the polynomials n_b and n_f are chosen such as to yield the same number of parameters as the linear model according to (6.29).

To each instance of the Monte Carlo simulation, the procedure following Algorithm 1 is applied. In this way, for each of the realisations an optimal lag for each behavioural model structure as well as the corresponding model is obtained. Based on the optimal lag of the linear kernel representation model, the corresponding EE model is identified. Both in the quantitative evaluation as well as in the resulting plots, the first 20 samples of the calibration and validation data set are not considered in order to let the models overcome their transient behaviour.

A semilogarithmic plot of the relative improvement δ_t over the lag t for both systems is shown in Figure 6.6. The curves indicate the mean, calculated over the Monte Carlo realisations, the error bars indicate the minima and maxima of the according value as shown by the Monte-Carlo simulation.

This plot shows a monotonically decreasing relative improvement for both models as well as narrowing limits when the lag is increased. According to the optimisation criterion, the optimal lag is determined for each Monte-Carlo run individually. For this purpose, the achieved relative improvement is compared to the specified minimum improvement $\delta_{min} = 10^{-3}$. The lag for each realisation is shown in Figure 6.7. Further this graph shows the stability of the identified models on calibration and validation data set. Unstable models are removed from the Monte Carlo study for the evaluation

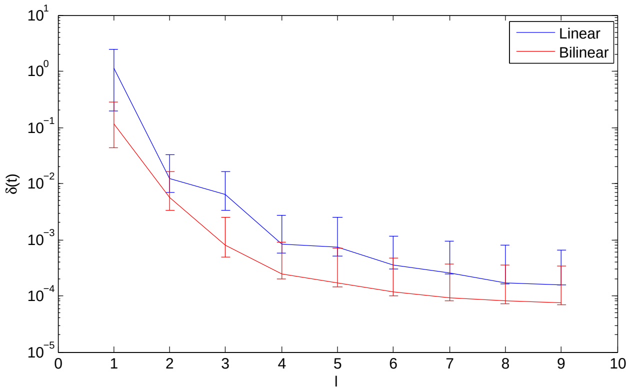


Figure 6.6: Logarithm of relative improvement $\log_{10} \delta_t$ over lag t for the respective model class (mean value and minima/maxima)

of the quantitative errors. Instability in this context is interpreted as 'not suitable for application' due to unbounded w_3, for this purpose a model exhibit $w_3 > 20$ is considered unstable.

From Figure 6.7, the fact that the lag structure of the model is selected according to the combined misfit/complexity criterion based on the data leads to a number of different models. These models are the best trade-off between arbitrarily good fit and manageable complexity.

The models as identified for each instance of the Monte Carlo simulation are simulated with w_1^0 imposed as port signal to port w_1 for both calibration and validation data set. One typical instance ($M = 500$) is shown in Figures 6.8 and 6.9 for calibration and validation data set, respectively.

From Figures 6.8 and 6.9, the improved simulation performance of the bilinear extended kernel representation from a qualitative point of view becomes evident. Owing to the shape of the tanh function, the nonlinear distortion acts mainly in the higher operating range, here the bilinear structure is able to express the behaviour appropriately. The linear kernel representation does not have the ability to follow these nonlinearities, neither does the EE structure. The former shows high errors towards the upper part of the operating range, the latter, perhaps by optimising the model on the additive error between both variables, shows a slightly better behaviour.

For a quantitative analysis of the modelling performance, the mean

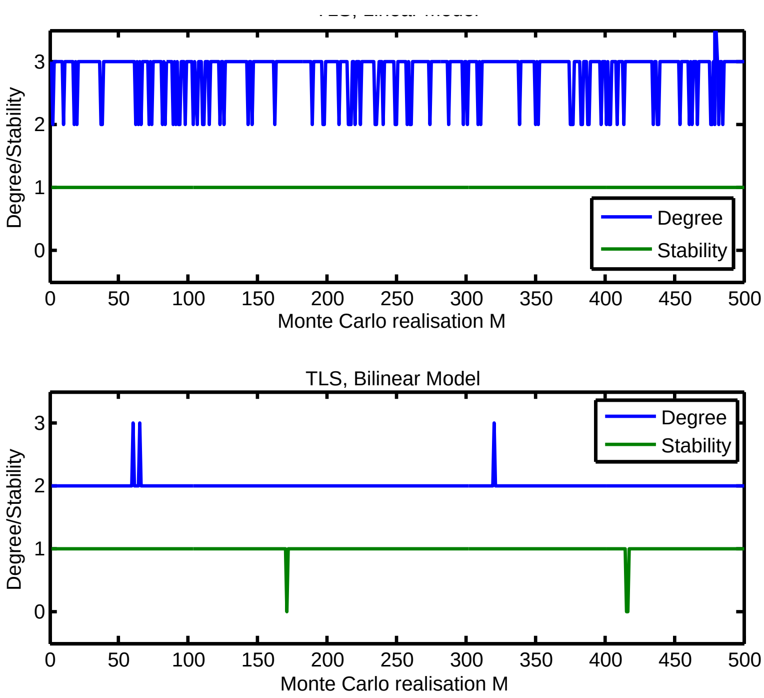


Figure 6.7: Lag (blue) and stability (green) of the individual Monte Carlo realisations, the green curve showing the value 1 indicates a stable model

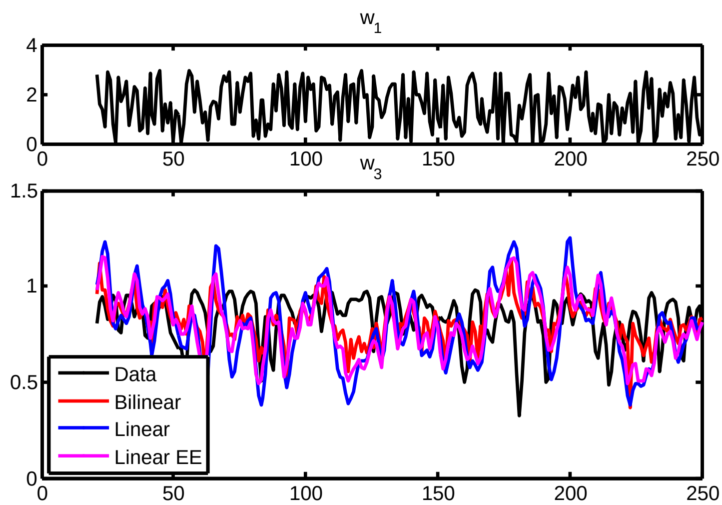


Figure 6.8: Example of the modeling performance on the calibration data set

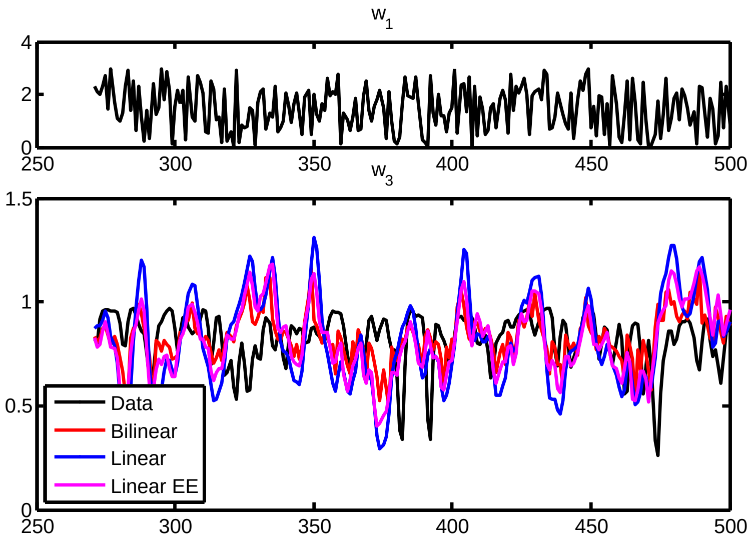


Figure 6.9: Example of the modeling performance on the validation data set

square error (MSE)

$$\mathrm{MSE}\left(\hat{w}_3\right) = \frac{1}{\|I\|} \sum_{k \in I} \left(\hat{w}_3(k) - w_3(k)\right)^2 \tag{6.35}$$

with the simulated port variable $\hat{w}_3$ and the appropriate index set I for calibration and validation data set is considered. For the Monte-Carlo analysis, the mean and variance of the MSE are analysed. The results of the quantitative analysis are presented in Table 6.1. The MSE indicates that the bilinear extended kernel structure is able to explain the data almost as good as the causal EE structure does, while it yields a significant improvement over the linear kernel representation when applied in this case. The improvement is achieved at the cost of a higher variance in modelling performance of the bilinear kernel structure as well as an increased number of unstable models.

The proposed algorithm for modelling in a combined misfit/complexity approach works well with the data present. The algorithm terminates and finds appropriate models, due to the variable-dependent stability of the bilinear model structure some of which are unstable.

Calibration	Bilinear Kernel	Linear Kernel	Linear EE
$\bar{\text{MSE}}$	0.0570	0.1349	0.0822
var (MSE)	0.0038	0.0102	0.0039

Validation	Bilinear Kernel	Linear Kernel	Linear EE
$\bar{\text{MSE}}$	0.0700	0.0911	0.0879
var (MSE)	0.0041	0.0047	0.0036

Table 6.1: Quantitative errors between Wiener system data and simulation results

6.5 Application study

6.5.1 Introduction

In order to further evaluate the applicability of the bilinear extended kernel representation, sampled data originating from a continuous time nonlinear model of a continuous stirred tank reactor (CSTR) is modelled. The algorithm introduced above is applied to model, with candidate models from two model classes, the classes of linear and bilinear extended kernel representations.

This forms an extension of the above numerical study since the model is simulated in continuous time and later sampled, while for the numerical study a discrete time model was selected. It further serves as a test of the applicability of the bilinear extended kernel structure and the combined misfit/complexity algorithm to practical systems.

6.5.2 Plant description

The CSTR model applied in this study is that of a constant volume reactor cooled by a single cooling jacket. The reaction in this reactor is an exothermic, irreversible reaction A $\rightarrow$ B. For the substances, perfect mixing is assumed. The model follows Henson and Seborg (1997) and is governed by

$$\dot{C}_a = \tfrac{q}{V}\left(C_{af} - C_a\right) - K_0 \exp\left(-\tfrac{E}{RT}\right) C_a \tag{6.36}$$

$$\dot{T} = \tfrac{q}{V}\left(T_f - T\right) - \tfrac{m}{\rho C_p} K_0 \exp\left(-\tfrac{E}{RT}\right) C_a + \tfrac{UA}{V \rho C_p}\left(T_c - T\right) \tag{6.37}$$

The coolant temperature T_c is considered as an external variable and is

State	Description	Initial Value	Unit
C_a	Concentration of A	0.99	mol/m^3
T	Temperature in CSTR	324.45	K
Parameter	Description	Value	Unit
q	Volumetric flowrate	100	m^3/s
V	CSTR Volume	100	m^3
ρ	Density of A-B Mixture	1000	kg/m^3
C_p	Heat capacity of A-B Mixture	0.239	J(kgK)$^-1$
m	Heat of reaction A $\rightarrow$ B	$5 \cdot 10^4$	J/mol
$\frac{E}{R}$	Arrhenius temperature dependence	8750	K
K_0	Pre-exponential factor	$7.2 \cdot 10^10$	s$^-1$
UA	Heat transfer	$5 \cdot 10^4$	WK$^-1$
C_{af}	Feed concentration	1	mol m$^-3$
T_f	Feed temperature	350	K

Table 6.2: States and parameters for CSTR simulation

used to excite the reactor. The remaining states and parameters are shown in Table 6.2.

Since the reaction A to B is an exothermic reaction in this model, it is not obvious to distinguish between input and output in this system.

Further to the ambiguous input/output structure of the system, the mutual dependence of both reactants leads to a nonlinear behaviour of the CSTR system. Figure 6.10 shows the steady state values of C_a of the system for different values of T_c, indicating the nonlinear steady state behaviour of the system. This figure also highlights that in the range considered, the model (6.36), (6.37) does not exhibit output multiplicities, which makes the class of bilinear models appropriate for this task. Indeed, the parameters chosen for this application study configure the CSTR model such that it does not exhibit output multiplicities over the entire operating range.

In addition to the fact that the CSTR model does not exhibit output multiplicities, it shows a nonlinear behaviour, as becomes obvious in Figure 6.11. In this figure, a series of steps, imposed on the coolant temperature T_c, shows the different magnitude of reaction of the other variables, clearly a sign of the nonlinear behaviour of the system.

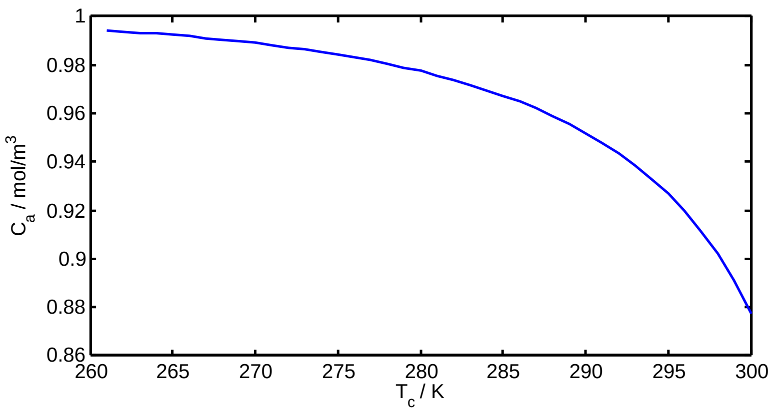


Figure 6.10: Steady state of C_a for different T_c levels

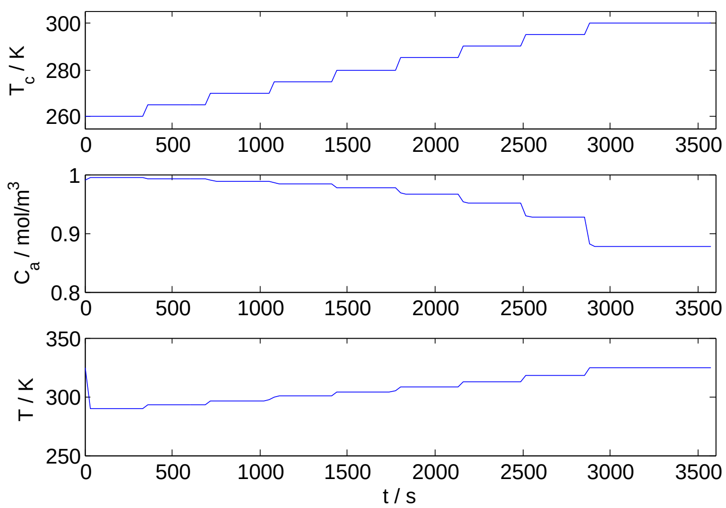


Figure 6.11: Series of steps excitation of CSTR model

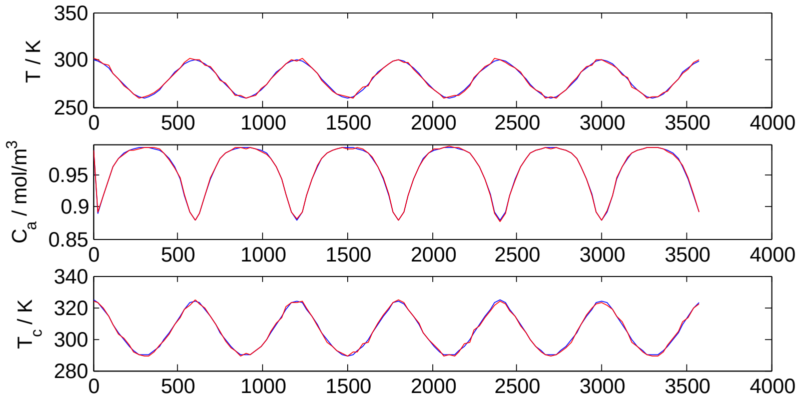


Figure 6.12: Full behaviour of the CSTR, noise-free and noise corrupted variables

6.5.3 Modelling

In order to obtain data for a test run of the modelling process, the CSTR model is simulated over a time range of 3600 s with samples being taken each 30 s. As an excitation signal, the coolant temperature $T_c = (280 + 20\cos\left(\frac{1}{300}\pi t\right)$ K is chosen. From this simulation, T_c and C_a are recorded with the aim of modelling the relation between both (thus considered as manifest variables), while T is considered a latent variable. The recorded data is corrupted by uniformly distributed white noise yielding a signal-to-noise ratio of 21.7 dB and 32.7 dB for T_c and C_a, respectively. The behaviour of the individual variables is depicted in Figure 6.12.

With the aim of fitting linear and bilinear kernel representations as described above, initially the correlation matrices according to (6.23), for both linear and bilinear signal spaces, are calculated. The misfit is related to the increase in complexity of the model structure as defined in Section 6.3.2. A plot of the relative improvement that can be achieved with linear and bilinear model structures is shown here in Figure 6.13, clearly indicating the advantage of the bilinear extended behaviour: for $l = 0 \ldots 9$, the bilinear extended behaviour promises to converge faster to its final modelling capacity.

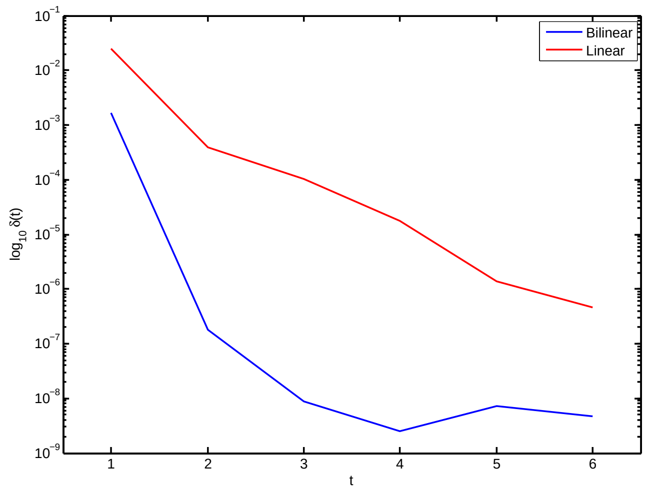


Figure 6.13: Relative improvement of the individual model structures plotted over order

Considering a required relative improvement of $\delta = 1 \cdot 10^{-4}$, the optimal order for the bilinear model is $l = 1$, while for the linear structure the algorithm returns $l = 3$.

The large initial improvement that becomes evident for an order $l = 1$ coincides with the known order of 1 of the system (6.36), (6.37) to be modelled in this example. Since the linear kernel representation is not able to accommodate the occurring nonlinearities, a higher order is required to terminate the algorithm by yielding the required relative improvement.

The linear kernel representation as in (2.37) with $l = 3$ is estimated by TLS from calibration data with $N = 2500$. The corresponding linear model is given by

$$\begin{bmatrix} 1.00 - 2.01\sigma + 1.28\sigma^2 - 0.215\sigma^3 \\ -0.000923 + 0.000132\sigma - 0.000105\sigma^2 - 0.00663\sigma^3 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \qquad (6.38)$$

A bilinear extended kernel representation as in (5.10) with $l = 1$ is

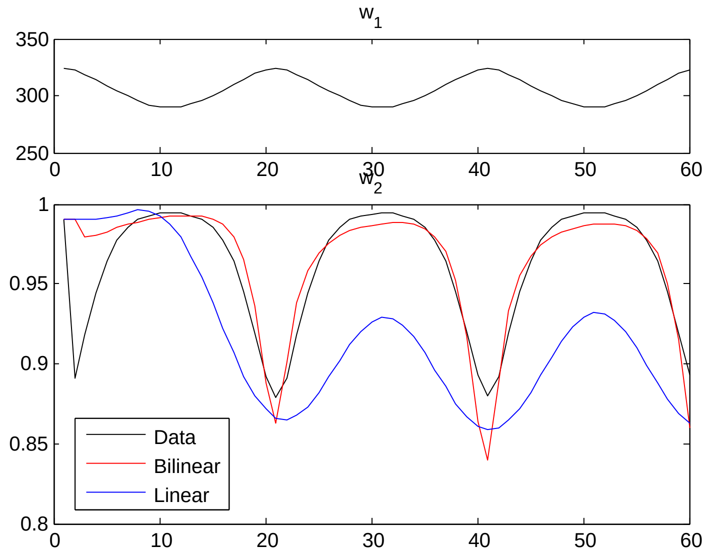


Figure 6.14: Modeling performance on calibration data set

estimated in the same way. The bilinear behavioural model is estimated as

$$\begin{bmatrix} 1.00 - 1.025\sigma \\ 0.00331 - 0.00332\sigma \\ -0.00330 + 0.00332\sigma \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \tag{6.39}$$

Here, w_3 is the bilinear term with $w_3 = w_1 w_2$.

The modelling performance of the models according to (6.39) and (6.38) on the calibration data set is illustrated in Figure 6.14. As could be expected from the steady state behaviour in Figure 6.10, the linear model structure is not able to explain the nonlinear distortion while the bilinear models explains it more appropriately.

6.5.4 Results

Both the linear and the bilinear model are simulated over $N = 1800$ time steps using the second part of the data from the simulation of the model (6.36), (6.37) for comparison. This part of the data is unseen by the modelling algorithm, the respective behaviour is shown in Figure 6.15. As for the

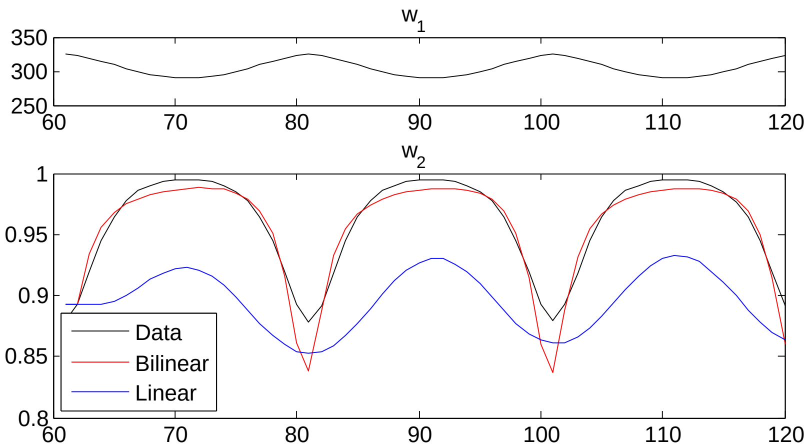


Figure 6.15: Modeling performance on validation data set

	Calibration		Validation	
	Bilinear	Linear	Bilinear	Linear
$\|\cdot\|_2$	0.1659	0.2513	0.1981	0.2738
$\|\cdot\|_\infty$	0.0552	0.0513	0.0548	0.0513

Table 6.3: Quantitative errors between CSTR data and simulation results

calibration data set, an inspection of the remaining misfit between model and data shows that a significant improvement may be achieved by application of the bilinear extended kernel representation. Especially in the higher temperature ranges, the linear model structure does not explain the features of the CSTR.

This qualitative impression is supported by the quantitative results given in Table 6.3. Here the model performance is assessed based on the Euclidean norm $\|\hat{C}_a - C_a\|_2$ and the maximum norm $\|\hat{C}_a - C_a\|_\infty$. As opposed to the Monte Carlo simulation where the mean improves while the variance develops unfavourably, the modelling improvement is achieved in all measured variables. However, since the considered case does not stem from a series of Monte Carlo simulations, repeated application of the algorithm to the data under different noise and excitation signals may produce outliers.

6.6 Concluding remarks

The inverse problem of finding the best model to explain a given data sequence is one of the key problems in systems and control and is of high practical importance. The present chapter develops a practical approach to approximate modelling. To this aim, the general process for modelling or system identification is expressed.

Within this general process, several techniques and paradigms can be applied. The common mainstream approach is incompatible with the postulate of the BF for an *a priori* symmetric treatment of the variables. To overcome the dominating part of this limitation, the EIV paradigm was developed and sees an increasing number of users. While this framework does not require an *a priori* definition of the input/output structure of a system, it is different from the parallel developments in the way misfit is interpreted.

While the common approach considers misfit to be modelled as a measurement error influence, the BF goes further in also assuming model mismatch, i.e. the case where the true system model is not contained in the set of candidate models. The modelling procedures for exact and approximate modelling are presented and the terminology is introduced, this serves as a basis for further development.

The case of approaching the modelling process with a set of candidate models not including the true system is relevant for application to practical systems, since in this case the expression of all governing equations will result in models that are too complex to be useful or provide insight into the system. While this puts the BF into a favourable position for the modelling of practical systems from data, the algorithms for modelling with limited complexity or limited misfit may result in sensitive models due to their strict optimisation of one property.

To overcome this drawback, the combined misfit/complexity algorithm, in a similar concept as proposed by Willems, is introduced. This algorithm considers a relative improvement of misfit related to the complexity increase and in this way resembles a heuristic approach of a control engineer seeking a robust model with handleable complexity and appropriate misfit. While this multivariable aim seems to be incompatible with optimisation, the combined misfit/complexity approach treating the relative improvement offers a way to achieve this aim.

The novel algorithm marks a contribution to the body of knowledge, as it extends the existing techniques for approximate modelling in the BF as described in (Willems, 1987) in a practical direction, as opposed to developments such as (Roorda, 1995) or (Markovsky *et al.*, 2005; Markovsky, 2006).

The novel algorithm is tested in a Monte Carlo simulation on two different acausal and one causal candidate model class. It is found that the algorithm forms a viable approach to the approximate modelling problem from the perspective of application, while it does not perform as well as the equation error method, it finds models of suitable complexity out of the sets of linear and bilinear extended kernel representations.

The further testing of the novel algorithm is executed in approximately modelling a CSTR based on sampled data stemming from a continuous time nonlinear model. As well as in the numerical study, the combined misfit/complexity algorithm operates well, resulting a suitable bilinear extended kernel representation expressing the nonlinear behaviour of the CSTR well.

The approach and algorithm developed in this chapter shows that the foundations of the BF are well compatible with applicable techniques for approximate modelling. The successful adaptation of these foundations to practical control problems makes the application to practical control problems more appealing.

The currently state of the proposed algorithm relies on the TLS estimator for misfit and parameter estimation. It is known that the TLS estimator is not we suited for dynamical systems due to the structural similarity of the lines of the observation matrix. A revision of the combined misfit/complexity algorithm, making use of current developments in the field of EIV identification, as given in (Söderström, 2007; Söderström *et al.*, 2002) may improve the results and the robustness.

Chapter 7

Towards adaptive control in the Behavioural Framework

A good scientist is a person with original ideas.
A good engineer is a person who makes a design
that works with as few original ideas as possible.
There are no prima donnas in engineering.

Freeman Dyson

7.1 Introduction

While the modelling task on its own is forms an interesting and well treated topic within the systems and control field, frequently the resulting models are employed for controller design or model based control.

In this chapter, a scheme for adaptive control by interconnection is developed and tested. The motivation for this stems from two sources. One is the possibility to obtain an increased energy efficiency by recuperation of energy for control purposes, the other is formed by the opportunities offered by interconnected controllers, for example applied to distributed power plants. While the BF and control by interconnection themselves both have appealing properties from a theoretical point of view, this work intends to show an approach closer to application.

The application of dissipative control by interconnection is considered as a means to increase energy efficiency by reusage of energy present in the systems for control purposes. Furthermore, dissipative control by interconnection may be expected to offer some inherent robustness features due to its architecture. This, in turn, may be considered as an advantage for remotely installed systems such as offshore wind generators or small water power plants. These devices operate to gain energy but need control of some kind to protect the systems.

The chapter starts with the introduction of control in the BF from a practical perspective. Based on this control paradigm, the problem of adaptive control is formulated and important differences to the feedback control scheme pointed out. As a practically viable solution, a scheme for adaptive control is developed based on control by interconnection and an EIV-KF/RLS tandem estimator proposed by (Larkowski, 2009).

In order to compare its performance to existing approaches, a comparative study with both controller and plant simulated in discrete time is conducted. The proposed setup for adaptive control is also tested on a continuous time plant in order to make it more closely resemble a practical setup.

7.2 Adaptive control in the Behavioural Framework

7.2.1 Control in the Behavioural Framework

For a model developed in the classical i/o framework, i.e. defining an input-output relation, the canonical choice for a control setup is the feedback control scheme. This paradigm was briefly addressed in Section 2.3.2 and illustrated in Figure 2.2. The main property of the feedback control paradigm is that it is based on the measurement of outputs and providing feedback to one or more input ports of the system, with this input being formed according to some control law. This paradigm is omnipresent in the technical world in many representations, as simple as the thermostat of a heating system, but also as complex as multidimensional autopilot systems.

While it is possible to design very effective feedback controllers for almost any purpose, many controllers present in everyday life do not act as feedback controllers in the sense of the feedback control paradigm. Simple passive controllers, such as heat fins, pressure control valves or shock absorbers (Trentelman and Willems, 2003) do not conform with the feedback control scheme. Most of these basic control systems could be replaced by a controller in the feedback control scheme, however the systems mentioned above serve the same purpose without additional control system.

Controllers that do not fit into the feedback control paradigm can be expressed in the causal model representation used, but to identify and express the controllers as such is difficult. Further, the feedback control scheme has no direct equivalent in the BF, as the strict causality - outputs are processed and fed back to the input - cannot be mapped to the BF. For these purpose, control needs to be addressed differently in the BF.

Control in the BF has been put forward in (Willems, 1997), while the textbook (Polderman and Willems, 1998), considered as an introductory text, mostly leaves the acausality of the BF when it comes to control. A comprehensive overview is presented in (Belur, 2003) and further publications (Willems *et al.*, 2003; Julius *et al.*, 2005; Belur and Trentelman, 2002; van der Schaft, 2003) work out details of control in the BF, such as regularity of interconnection and achievable behaviours of systems controlled by interconnection.

The central idea behind control in the BF is the interconnection of a plant and a controller along some ports, in order to make the interconnected behaviour the desired one. The variables to be controlled are denoted by w and those accessible to control by c. The controller also provides a port c, which is compatible to the plant in terms of time axis and signal space.

Before interconnection, the variables w and c satisfy the plant or the controller behaviour, respectively. By interconnecting plant and controller according to Section 2.4.3, c has to satisfy the behaviour of plant and controller simultaneously. This ensures that the laws are transferred to the plant.

The problem of control in the BF is thus reduced to that of finding an appropriate system to interconnect the plant to, which may be specified in some suitable function space according to (van der Schaft, 2003).

Given a plant Σ_p and a controller Σ_c , i.e.

$$\Sigma_\mathrm{p} = (\mathbb{T}, \mathbb{W}_w \times \mathbb{W}_c, \mathcal{B}_\mathrm{p}) \tag{7.1}$$

$$\Sigma_\mathrm{c} = (\mathbb{T}, \mathbb{W}_c, \mathcal{B}_\mathrm{c}) \tag{7.2}$$

with a common time axis and subset of the signal space, the interconnection of the systems can be formulated as

$$\Sigma_\mathrm{f} = \Sigma_\mathrm{p} \wedge \Sigma_\mathrm{c} = (\mathbb{T}, \mathbb{W}_c, \mathcal{B}_\mathrm{f}) \tag{7.3}$$

with the full behaviour given by

$$\mathcal{B}_\mathrm{f} = \left\{ (w,c)^T : \mathbb{T} \to \mathbb{W}_w \times \mathbb{W}_c \middle| (w,c)^T \in \mathcal{B}_\mathrm{p} \wedge c \in \mathcal{B}_\mathrm{c} \right\} \tag{7.4}$$

The controlled systems behaviour $\mathcal{B}$ relates to the manifest behaviour of the interconnected system, which is defined as

$$\mathcal{B} = \left\{ w : \mathbb{T} \to \mathbb{W}_w \middle| \exists c : \mathbb{T} \mapsto \mathbb{W}_c : (w,c)^T \in \mathcal{B}_\mathrm{p} \wedge c \in \mathcal{B}_\mathrm{c} \right\} \tag{7.5}$$

The paradigm of control by interconnection does not only differ from the feedback control scheme in terms of the incorporation of the controller in the system. Additionally, one major change can be observed in the choice of control objectives. In a framework based on model formulations without choice of input or output, the classical control targets such as disturbance rejection or setpoint tracking would lead to an *a priori* distinction between the variables.

To illustrate this paradigm of control and a related control target, a mass-spring system as the plant is considered. This system exhibits sustained oscillation after excitation by a force, which will be regarded as undesired behaviour. The control objective is to find a controller to be interconnected in order to stop the system from oscillating in the position variable.

Example 6 (Control by Interconnection) *The plant is a dynamical system $\Sigma_{\mathrm{p}} = (\mathbb{T}, \mathbb{W}, \mathcal{B}_{\mathrm{p}})$ with*

$$\begin{aligned} \mathbb{T} &= \mathbb{R}^{+} \\ \mathbb{W} &= (x, F_e, F_c)^T \subseteq \mathbb{R}^3 \\ \mathcal{B}_{\mathrm{p}} &= \left\{ w : \mathbb{T} \mapsto \mathbb{W} \middle| m\ddot{x} + cx = F_e + F_c \right\} \end{aligned} \tag{7.6}$$

In (7.6), x is the position of the mass, F_e an external foce and F_c denotes a control force. The uncontrolled behaviour of the plant, with $F_c = 0$, is set to exhibit a sustained oscillation.

In order to be able to reach the control objective, a control force F_c has to be applied to the mass by a controller yet to be defined. In the feedback control scheme, this controller would be formed by an actuator, controlled by a feedback controller. This feedback control system then measures the position, processes this variable according to some control law and operates the actor according to this.

Interconnected control instead would search for a system to be interconnected to the plant. This controller may be a viscous damper with damping coefficient b, governed by the differential equation $F_c = -b\dot{x}$, yielding the controller behaviour

$$\mathbb{W} = (x, F_c)^T \subseteq \mathbb{R}^2 \mathcal{B}_{\mathrm{c}} = \left\{ w : \mathbb{T} \mapsto \mathbb{W} \middle| F_c = -b\dot{x} \right\} \tag{7.7}$$

The full behaviour of the interconnected system is thus

$$\mathcal{B}_{\mathrm{f}} = \left\{ w : \mathbb{T} \mapsto \mathbb{W} \middle| m\ddot{x} + cx = F_e + F_c \wedge -b\dot{x} = F_c \right\} \tag{7.8}$$

This behaviour consists of two manifest variables, x and F_e, and one latent variable F_c.

By elimination of F_c, it is possible to state the manifest behaviour as

$$\mathcal{B} = \Big\{ w : \mathbb{T} \mapsto \mathbb{W} \Big| m\ddot{x} + b\dot{x} + cx = F_e \Big\} \tag{7.9}$$

which represents an oscillator controlled by interconnecting a damper to yield, for instance, a critically damped mass-spring-damper system.

This example relates to the interconnection of a damper to a non-dissipative oscillator, which in the strict notation of the BF is developed as in this example. While this example was chose to be obvious in order to allow for a comparison, more complex systems and controllers may require appropriate notation to become approachable.

As becomes obvious from the differential control action of the viscous damper, the feedback controller performing the same control may suffer from noise amplification, especially since the position information stems from measured data. In comparison to this, the resulting controlled system in the control by interconnection paradigm, a mass-spring-damper system, is known to behave well.

7.2.2 Problem statement

The presence of a paradigm for control in a behavioural context, limited to time-invariant (LTI) systems, calls for an extension to encompass linear time-varying (LTV) and nonlinear systems. This is important for application, since in may application examples, the assumption of a systems being LTI is violated. While the latter type of systems may be controlled effectively by nonlinear control techniques, both types are accessible to adaptive control techniques, depending on the degree of nonlinearity in the case of nonlinear systems.

Indeed, the step towards adaptive control in the BF is not new. The possibility to exploit the novel way of behavioural system modelling for adaptive control was first mentioned in (Willems, 1986b), where the application of the MPUM for adaptive control was proposed. This line is followed throughout the publications on the subject of adaptive control in the BF, such as (Polderman, 2000; Polderman and Mareels, 1999).

The problem to be solved is based on a plant $\Sigma = \{\mathbb{T}, \mathbb{W}, \mathcal{B}_p\}$ with a continuous time axis $\mathbb{T} \subseteq \mathbb{R}$, a signal space $\mathbb{W} = \mathbb{W}_p \times \mathbb{W}_c$, accessible for control through $\mathbb{W}_c$ and a plant behaviour Σ_p. This plant behaviour can be of LTV nature, resulting in a latent variable formulation with time-varying matrix ploynomials R and M given by

$$\Sigma_p = \left\{(w,c) \in \mathbb{W}^{\mathbb{T}} | R\left(t, \frac{d}{dt}\right) w = M\left(t, \frac{d}{dt}\right) c\right\} \tag{7.10}$$

In this context, let R and M denote matrix polynomials in $\frac{d}{dt}$ with time varying coefficient matrices in $\left(\mathbb{R}^{g\times q}\right)^{\mathbb{T}}$.

Further the plant behaviour may be of a more general nonlinear nature, that does not allow a representation as in (7.10) and can be expressed as

$$\Sigma_p = \left\{(w,c) \in \mathbb{W}^{\mathbb{T}} | r\left(w, t, \frac{d}{dt}\right) = m\left(c, t, \frac{d}{dt}\right)\right\} \tag{7.11}$$

where r and m denote functions $\cdot : \left(\mathbb{W}^{\mathbb{T}}, \mathbb{T}, \frac{d}{dt}\right) \to \mathbb{R}$ expressing the nonlinear differential equation.

Both R or r are assumed to be unknown. The problem is to find a suitable time varying polynomial matrix or function to achieve the given control target.

This problem can be subdivided into smaller subproblems, for sake of simplicity only the LTV case (7.10) is treated explicitly:

- Define a control target
- Find the appropriate model class $\mathcal{R}$ for R
- Express the controller class $\mathcal{C}$ for C
- Select the best model for each time step $\hat{R}$
- Based on $\hat{R}$ and the control target, find the controller C that achieves the desired behaviour
- Update the controller at each time step

The above list of subproblems highlights some practically encountered limitations, these are the finite sampling time of digital computers and the preselection of the controller structure. The latter is necessary since a practical controller will have to be integrated into the system and consequently cannot change its structure easily. All other subproblems exhibit a difference from the classical adaptive control approach that is addressed in the sequel.

Control targets

In an input/output setting, the common control targets are mostly based on an input-output comparison (Albertos and Mareels, 2010), such as:

- Regulation or disturbance rejection: This control objective aims to keep one or more of the variables at a constant level, while other variables, latent or manifest, experience disturbing effects.
- Tracking: In this case, the control objective is to follow a given signal as closely as possible, with the assumption of variables being subject to disturbances.
- Optimisation: here the control system aims to optimise the system variables with respect to some target function instead of relations between single variables.

In a more global view, considering the system to be controlled as made up of subsystems or being more than just a signal processor, more control objectives exist (Albertos and Mareels, 2010):

- Adaptation: the aim of adaptation is to maintain an overall system behaviour despite changes in the system behaviour.
- Fault detection and process reconfiguration: here the control system avoids unsafe or undesired operation by either alarming operators or taking automatic countermeasures.
- Supervision: the objective of supervisory control is to monitor a multilevel control structure and to decide on the control targets for the subordinate control systems.
- Coordination: here the aim is to provide lower level control systems with the appropriate control target to ensure operation of processes that are controlled by local control systems.

In an acausal framework the well known control objectives, which are mainly based on comparison between dedicated input and output, as given in the first list, have limited applicability. A suitable control objective for the BF needs to specify in a symmetric way the membership of the time trajectories to the controlled behaviour (Willems, 1997). In this sense, the control objectives in the second list form an appropriate subset, especially the objective of adaptation to a given desired behaviour is well compatible with the BF.

Plant model class

When attempting to model a system in the BF for adaptive control purposes, it is most natural to apply a linear kernel representation as in (2.37). This model class does not introduce causality *a priori* and for the linear subset of behavioural models, control techniques and stability measures exist.

However, as indicated above, it is very likely that the controller applied in the practical system under study is not flexible in its behavioural structure, e.g. an adaptive damper will always be a first order differentiating controller. For this reason, the controller cannot fully express the optimal control for arbitrary plants.

Two solutions for this problem are suitable, on the one hand, one could use modelling techniques that vary the structure by increasing the lag of the structure (modelling with limited misfit or similar techniques) and crop the resulting model. On the other, it is possible to model with a fixed structure (a variation of the modelling with limited complexity approach) and assume the misfit to be caused by model mismatch. This latter approach appears to be the more natural and in fact is the approach frequently chosen in applications.

Controller model class

When applying adaptive control to practical systems, the interconnected controller has to be technically implementable. This means that it must be possible to assemble it from a finite number of basic technical elements, consequently the controller model class cannot be chosen freely.

Instead the controller model class is restricted by the availability of controller implementations and while in theory it is possible to switch or blend between different controllers, this is not typically in the range of practical applications. This model class limitation due to practical and implementation reasons almost inevitably leads to a grey box approach, in which the controller and plant model class is selected by making use of analytical modelling of the system while the model selection and parameter estimation is performed by the control algorithm.

Plant model selection

One of the main tasks in adaptive control is the online selection of an appropriate model representation of the plant behaviour by an appropriate identification algorithm. Here special attention needs to be paid to the acausality assumption, which leads to an inability to specify an input/output structure. Consequently, all variables have to be assumed to be subject to noise disturbances. At the same time, a suitable identification procedure has to run online, while most EIV-identification techniques are design to run in batch operation and are thus inherently unsuitable for adaptive control.

Further it is important to point out the fact that a model of a given behaviour is not unique, but merely forms one representation. Within one model class, certain normalisations need to be performed in order to obtain a suitable basis for controller selection.

Controller selection

Based on the selected model for the observed plant behaviour and the control target, a controller has to be found. Depending on the control target formulation, this requires analytical calculation or perhaps optimisation within the class of controllers.

Controller update

Whereas in the in the controller selection step, the theoretically optimal controller according to the control target is determined, depending on the application and operating condition, frequently an immediate attainment of this controller is not feasible or desirable, e.g. from the point of view of wear of actuators. While this may be included in the control target, one may alternatively include additional update stages in the procedures.

7.2.3 A practical approach to adaptive control in the BF

Preliminaries

The approach outlined in this section provides a practically applicable answer to the problem of adaptive control in the BF and its subproblems.

As the solution aims to be applicable to practical systems, the following conditions are assumed:

1. Measurement noise on all manifest variables
2. Latent variables are inaccessible for measurements
3. Use of a digital computer control system for model selection and controller update
4. Use of a parametric interconnected control element with the time axis of the plant

Condition 1) reflects that the acausal nature of the BF leads naturally towards an EIV formulation since no input/output partition can be assumed beforehand. Condition 2) forms a common assumption in system identification, but also forms the first advantage of interconnected control. Since the interconnected controller is connected to the variables in the controller behaviour, regardless of manifest or latent from a modelling point of view, it does not have to rely on observed variables for action on these.

With postulating condition 3) for this study, a causal subsystem in the acausal control system is formed. This causal subsystem is due to the signal processor nature of the digital computer control system, which needs a finite processing time for all tasks. This causal subsystem could be circumvented by use of an analog computer, however this would reduce the practical applicability. Further the causal subsystem does not impact the acausal operation of the overall system and thus can be accepted for applications.

In condition 4), a control element with fixed dynamic structure that can be changed in its parameters only is postulated, together with the reasonable assumption that it operates on the same time axis as the to be controlled system. This avoids having to interconnect subsystems with different time axes, instead a time-varying system with the same time axis is interconnected.

Plant and controller model class

The plant model class is a subset of the class of kernel representations, which is narrowed down further by analytical modelling of the plant in question. All models are linear time invariant in nature, but are updated iteratively, which results in time varying models.

As an effect of this selection of the plant model class, the modelling procedure only has to bear misfit due to measurement noise and model mismatch caused by the different behaviour of time varying systems and by neglected plant behaviours in the analytical modelling step.

The controller model class is similarly derived, a subclass of the kernel representation class is specified following analytical modelling of the controller. Also here, a slight model mismatch may be expected, also caused by the different behaviour of time varying systems compared to LTI systems and eventually neglected behaviours.

The mismatch between the plant or controller and their respective model classes due to the LTI assumption can be considered small compared to the effect of the time varying plant behaviour, since either this time varying behaviour is varying quickly, leading to an increased effect, or it is varying slowly, reducing also the effect of the LTI assumption of the models.

Plant model selection

Out of the class of plant models $\mathcal{R}$ the time dependent model $R(t) \in \mathcal{R}$ has to be selected at each sampling instant. While the behavioural equations of a given behaviour is not unique and consequently the model selection is not bound to find a unique representation, the behaviour of a given model is unique. For this reason, the intermediary process step of finding a model representation is necessary since the predefined model structure helps to express the dominant behavioural features.

The problem of recursive plant model selection can be addressed by a recursive EIV system identification technique such as the tandem setup consisting of an Errors-In-Variables Kalman Filter (EIV-KF) and a Recursive Least Squares (RLS) estimator as proposed by (Larkowski, 2009). This tandem scheme works as follows: the noisy sampled observations of the manifest variables of the system $w^0 + \tilde{w}$ is used by the EIV-KF, to yield an estimate of the noisefree manifest variables $\tilde{w}_0$, under consideration of the symmetric noise scenario present in the system. These estimated manifest variables are then forwarded to the RLS, where a set of parameters for the model class under consideration $\hat{\theta}$ is estimated, i.e. a model is selected. These parameters are returned to the EIV-KF for an update of the system matrices used in the KF. In this way, a self-contained online estimator is generated that is able to provide the adaptive control system proposed with a model, based

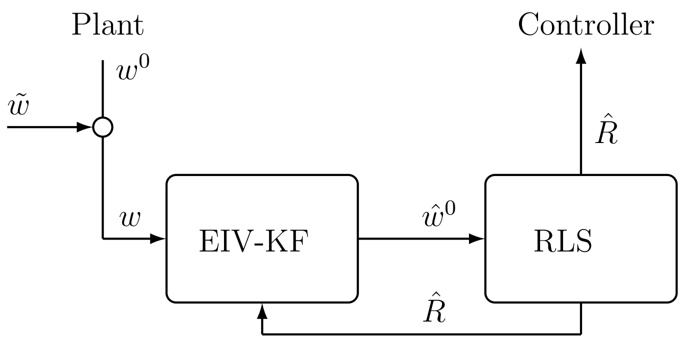


Figure 7.1: Recursive plant model selection setup consisting of EIV-KF and RLS

on a recursive update of the behavioural equations following changes in the behaviour and taking into account measurement noise on all variables. This scheme for recursive model selection is depicted in Figure 7.1.

In the original work (Larkowski, 2009), an additional estimation stage for an initial determination of the system matrices of the EIV-KF is used. This is not carried out in the present work, instead the EIV-KF is initialised with the true parameter set before occurrence of any changes in the parameters. Such a reference parameter set can be obtained in practical applications in many cases, e.g. during commissioning phase.

Controller model selection

Out of the class of controller models, the controller that guarantees an overall behaviour as close as possible to the desired one is selected for the time dependent controller model. This selection is based on the current plant model and the desired behaviour. The plant model is processed to obtain the behavioural features that are necessary for comparison to the desired behaviour, depending on the type of control objective.

In the controller model selection process step, it is important to optimise within the possible set of controller behaviours, e.g. when an adaptive resistor is available as an interconnected controller, it is not acceptable to select a general impedance with capacitive and inductive parts as a controller.

Due to the plant model selection based on the manifest variables only and the full integration of the controller into the plant, the controller behaviour is part of the selected plant model. This can be compensated for by iteratively updating the the controller behaviour.

Controller update

The adaptation of the controller to the plant model, since it is executed at discrete time steps according to the conditions assumed for this study, effectively constitutes a multiple model structure with local models selected based on the current behaviour. As shown in (Branicky, 1994), these systems may exhibit unstable behaviours even if the local models show stable behaviours. Introducing a smooth transition between the local models, i.e. blending between the structures, may circumvent these problems, as indicated in (Pfaff, 2006).

Further reasons for a preprocessing step before atttaining the controller assumed to be optimal arise from the general physical limitations of real-world systems, e.g. the above mentioned adaptive resistor may not be suitable for the whole real plane of possible voltage-current-tuples, but instead may only serve a subset of limited power.

For this reason, given the aim of practical applicability of the proposed method, an optional controller update step is set aside for these cases.

The overall scheme is depicted in Figure 7.2, the related data flow scheme is shown in Figure 7.3. Here a $\star$ marks optional processing steps, which can be omitted for discrete time systems or for systems not requiring a preprocessed controller update.

This scheme makes no assumption on the input/output structure of the to-be-controlled system, making it especially useful for systems where input and output cannot be selected uniquely or where energy and signal flow reverse during the operation of the system and the controller. Further, thanks to the interconnection of plant and controller, the controller becomes part of the system. Under certain circumstances, e.g. differentiating controllers, this may improve robustness, as pointed out in (Willems, 1997) for a door closing mechanism.

The scheme developed represents a practically feasible approach to adaptive control in the BF, a problem that has not been addressed in this way in the literature, to the best knowledge of the author. It differs from the approach chosen by (Polderman and Mareels, 1999) and related works such as (Safonov, 2001) in that it takes into account limitations of practical control

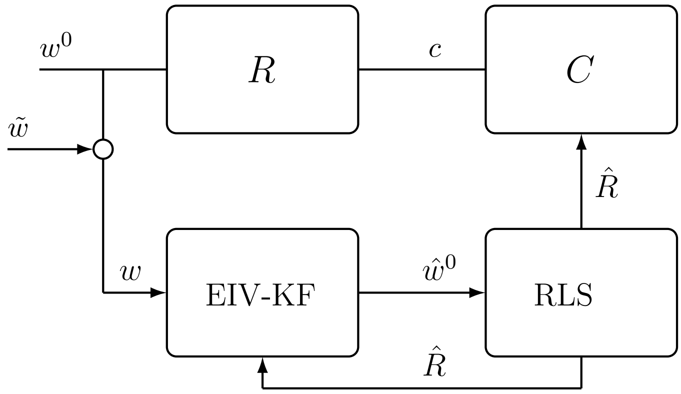


Figure 7.2: Setup 3: EIV-KF/RLS with control by interconnection

systems, such as a finite processing time and noisy measurements.

The scheme makes use of the EIV-KF/RLS tandem estimator in a slightly different form for adaptive control, which extends the work of (Larkowski, 2009). The scheme is implemented and tested in Simulink S-functions (as described in Appendix B), which makes them principally ready for porting to a practical control system. Indeed the simulations presented in Sections 7.4 and 7.5 demonstrate the practical nature and implementation readiness of the control system.

The newly developed scheme is subjected to a two stage test: initially it is compared to two alternative adaptive control schemes in a Monte Carlo study using a discrete-time simulation. In the second stage the scheme is tested on continuous time models of linear time varying and bilinear behaviours.

7.3 Comparative study

7.3.1 Preliminaries

The newly developed scheme is compared to two different schemes, which represent an increasing degree of acceptance of the BF for modelling and control. The increasing degree of acceptance of the BF may be manifested

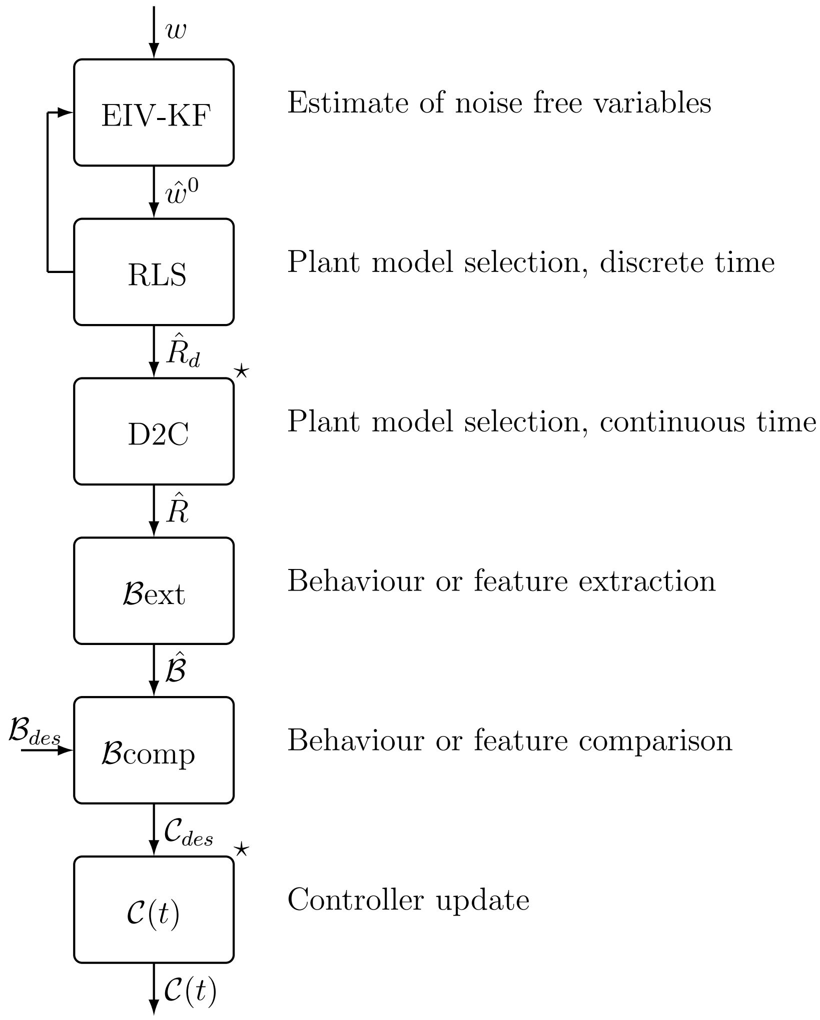


Figure 7.3: Data flow diagram of the proposed adaptive control by interconnection scheme, $\star$ denotes optional steps

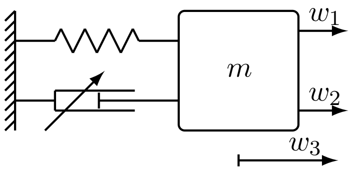


Figure 7.4: Mass-spring-adaptable damper system

in the symmetric treatment of the variables, leading to an EIV formulation, or in the conceptualisation of a controller as an integral part of the system, leading to an interconnected control approach.

The three scenarios considered are, in ascending order of acceptance of the BF, the following

1. Model selection by Kalman filter (KF), velocity dependent feedback control
2. Model selection by KF, control by interconnecting an adjustable damper
3. Model selection by EIV-KF/RLS tandem scheme, control by interconnecting an adjustable damper

Plant model class

The system under consideration is a discrete time formulation of a mass-spring system, shown in Figure 7.4, with a corresponding continuous time formulation given in (7.6). At a sampling interval $T_s = 0.5$s and for $m = c = 1$, (7.6) relates to a discrete time equivalent

$$\begin{aligned} \mathbb{T} &= \mathbb{Z}^+ \\ \mathbb{W} &= (w_1, w_2, w_3)^T \subseteq \mathbb{R}^3 \\ \mathcal{B}_{\text{plant}} &= \Big\{ w : \mathbb{T} \mapsto \mathbb{W} \Big| \\ 0.122\left(\sigma + \sigma^2\right)&(w_1 + w_2) + \left(1 - R_{3,1}\sigma + \sigma^2\right) w_3 = 0 \Big\} \end{aligned} \tag{7.12}$$

where $R_{3,1} = -1.76$, w_1 denotes the external force, w_2 the control force and w_3 the position of the mass.

In order to test the adaptivity of the proposed algorithm in the simulation of a degrading system, the spring constant $c(t)$ is assumed to vary over time as

$$c(t) = \begin{cases} 1 & , t \leq 10 \\ 1 + 0.05(t - 10) & , t > 10 \end{cases} \tag{7.13}$$

This time varying parameter relates to a changing discrete time parameter defined by

$$R_{3,1}(k) = \begin{cases} -1.76 & , k \leq 20 \\ -1.76 + 5.75 \cdot 10^{-3}(k - 20) & , k > 20 \end{cases} \tag{7.14}$$

Thus for the model selection stage, the suitable model class is that of lag 2 kernel representation models, such that in the model selection stage no causality assumption is introduced.

Control objective and controller model class

From an inspection of (7.6), it becomes obvious that the system will exhibit sustained oscillation after excitation, an undesired behaviour in many applications. The addition of a damper to the system would in general suffice to stop excessive oscillations, however a constant damper may lead to either over- or underdamped behaviour, which in many applications is suboptimal.

The control objective is to add a variable damper to the system in order to maintain critical damping under variation of the spring constant, so that the controller is formulated as in (7.7), which may be related to a discrete time equivalent

$$\begin{aligned} \mathbb{T} &= \mathbb{Z}^+ \\ \mathbb{W} &= (w_2, w_3)^T \subseteq \mathbb{R}^2 \\ \mathcal{B}_{\text{controller}} &= \left\{ w : \mathbb{T} \mapsto \mathbb{W} \middle| w_2(k) = -b \frac{w_3(k) - w_3(k-1)}{T_s} \right\}. \end{aligned} \tag{7.15}$$

The controller model class under consideration is thus a subspace of all lag 2 kernel representations, able to express the family of dampers as described in (7.15).

Model and controller selection

KF and feedback control This setup is related to the lowest degree of acceptance of the BF, which is expressed in the fact that neither an EIV model selection technique nor an interconnected controller is assumed. Since no capability for EIV estimation is required in this scheme, a KF configured to estimate the parameters of a second order ARX system according to the KF algorithm presented in (Ljung, 1999).

The KF is operated in a single-input-single-output (SISO) manner, as the control force w_2 is generated outside the system and thus only the resulting force $w_{1,2} = w_1 + w_2$ excites the system. Both measured variables, the exciting force w_1 and the measured position w_3, are subject to measurement noise; while the controller feedback w_2 is assumed to be known, i.e. noise-free. The KF is initialised with the true parameter set and the error covariance matrix $R_w = 0.01I$.

Based on the estimate of $R_{3,1}$, the necessary damping coefficient is emulated by a feedback control loop according to the control law

$$w_2(k) = b(k)\frac{w_3(k) - w_3(k-1)}{T_s} \tag{7.16}$$

with $T_s^{-1}b(k)$ calculated as

$$\frac{b(k)}{T_s} = \hat{R}_{3,1}(k-1) - 1.21 \tag{7.17}$$

Based on an estimate of the system parameters, this control law ensures slightly more than critical damping.

Noise is an artefact that is not inherently present on the system variables. In the case of an interconnected controller, this is of particular importance. Since the controller in the BF is interconnected to the system via the system variables, it is not affected by noise. In the case of feedback control generated by an feedback control law, the controller is subject to measurement noise, since it has no access to the noise-free variables.

KF and behavioural control This scheme indicates a further-reaching acceptance of the BF in that an interconnected controller is selected to achieve the control objective, however the KF estimator is not well suited for the EIV estimation problem at hand. This choice however may appear reasonable when taking into account that recursive EIV state or parameter estimation techniques are relatively young and little used in practice.

Since in contrast to the feedback controller devised above, the control force w_2 is unknown in this scheme, a force transducer is assumed to measure this variable, which contradicts with assumption 2) of this study. The noisy measurement signal serves as one of the variables for parameter estimation, thus the KF is operated in a multiple-input-single-output (MISO) setup. It is initialised with the true initial parameter set and the error covariance matrix is set to $R_w = 0.01I$.

The parameters estimated are used to select the appropriate controller out of the controller model class defined in (7.15) with the parameter b selected according to (7.17).

EIV-KF/RLS tandem setup and adaptive control This setup, which takes into account noise on all measured variables, assumes an acausal model structure and uses an interconnected controller, is likely to mark the true behaviourists choice. An EIV-KF/RLS tandem setup is used to estimate the parameters of a lag 2 kernel representation, which does not introduce any *a priori* assumptions on the causality, neither by considering noise free input nor by defining a causal model structure for the parameter estimator.

The EIV-KF applied in the numerical study follows the description given in (Diversi *et al.*, 2003; Guidorzi *et al.*, 2003), including the feedforward term to ensure acausality of the underlying model. Both the EIV-KF and the RLS use a second order discrete time kernel representation.

The EIV-KF is initialised with the true initial parameter set as well as the true noise variances and covariances. After a convergence time of 10 s, the system parameters are supplied by the RLS estimator, before this, the true parameter set is used without update.

The noise-free variables $(\hat{w}_1^0, \hat{w}_2^0, \hat{w}_3^0)^T$ estimated by the EIV-KF are used by the RLS estimator, which on its own is not capable of delivering unbiased estimates in an EIV noise situation, to yield unbiased parameter estimates. After twenty time steps, the parameters obtained are used by the EIV-KF and the adaptive control algorithm. The estimator is initialised with a covariance matrix $P_0 = 0.5I$ and it is configured to adapt to parameter changes with a fixed forgetting factor $\lambda = 0.95$.

The adaptive controller is interconnected to the variables w_2 and w_3 according to the controller behaviour (7.15).

7.3.2 Simulation

A numerical study, with the system as defined in Section 7.3.1, is conducted to investigate the applicability of the approaches and identify potential advantages and drawbacks. This study controls the system being subjected to an excitation generated by

$$w_1(k) = \mathcal{U}_{[-0.1,0.1]} + \begin{cases} 1, k \mod 50 < 25 \\ -1, k \mod 50 \geq 25 \end{cases} \tag{7.18}$$

with $\mathcal{U}_J$ denoting uniform distributed noise in a given interval J.

As the BF does not state a system's causality *a priori*, measurement noise in the form of Gaussian distributed white noise is added to all three variables (except for the KF/feedback control scheme), yielding a signal-to-noise ratio of 28 dB for w_1, 18 dB for w_2 and 36 dB for w_3.

A Monte Carlo simulation comprising 100 runs is conducted with the three schemes running in parallel on their respective system setup, with the noise sequence and the excitation signals being the same.

7.3.3 Results

The three control schemes are assessed according to their ability to achieve the system behave as desired and, as an indirect measure, their ability to estimate the parameter $R : 3, 1$ which is used for determination of the damping coefficient.

Figure 7.5 shows a comparison between w_1^0 and the true and estimated position, i.e. w_3^0 and $\hat{w}_3^0$, respectively. The data originates from a randomly chosen yet typical realisation of the Monte-Carlo study. From a qualitative point of view, the control scheme 3, i.e. EIV-KF/RLS and behavioural control, reaches the control aim the closest, control scheme 2, i.e. KF and behavioural control, after a long time of transient behaviour resembles the result of control scheme 3 and control scheme 1 shows constant oscillation.

Figure 7.6 illustrates the parameter estimation performance of the three applied estimators on the parameter $R_{3,1}$, which is used for determination of the adaptive damping coefficient. Here the parameter estimates originating from scheme 1 show dependence on the input signal, but in average follow the true parameter rather closely, while the estimates from scheme 2 seem

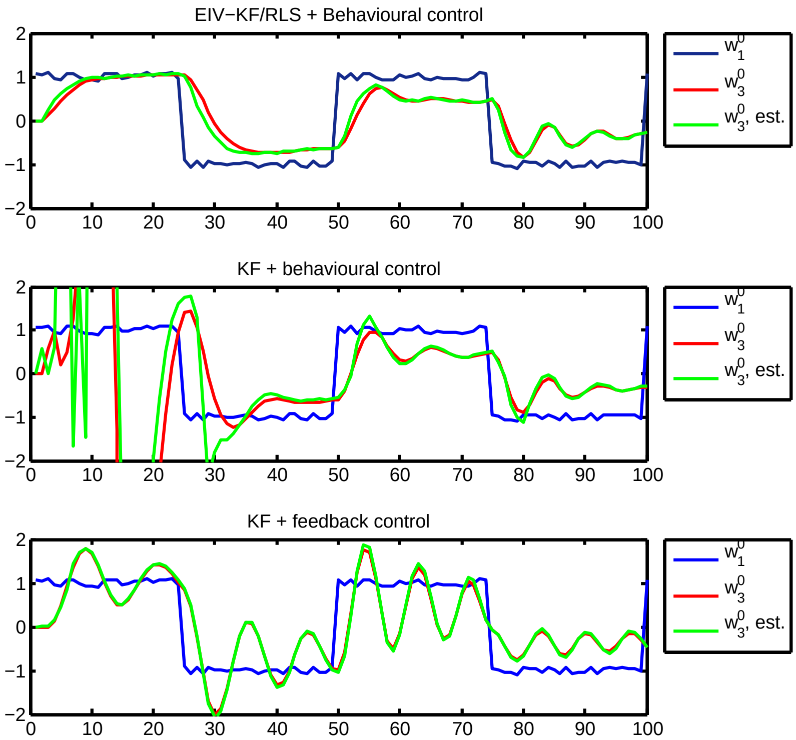


Figure 7.5: True w_1 and estimated as well as true w_3 of the single approaches

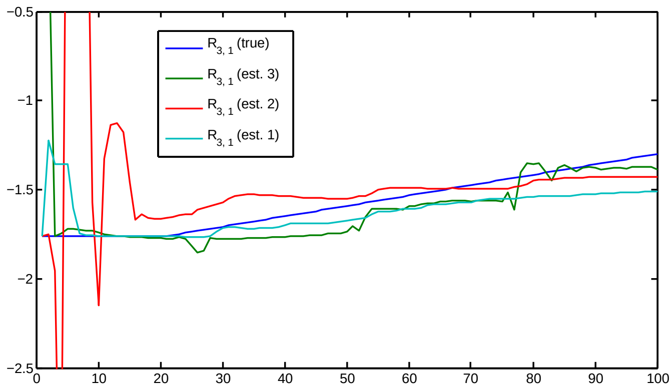


Figure 7.6: True and estimated parameter $R_{3,1}$ of the single approaches

to be influenced by the symmetric noise scenario and the estimates from scheme 3 also seem to converge to a similar biased value.

From a quantitative point of view, the difference between the desired and achieved behaviour can be analysed by making use of mean and variance of $w_3^0 - w_{3,des}$, i.e. the difference in position between the actual and a critically damped system. This error marks the misbehaviour of the current controlled plant against the desired one, as visible from the manifest variable w_3^0. This error shall be as low as possible and vary as little as possible over the Monte Carlo iterations.

These results are shown Table 7.1. In this table, the qualitative impression is generally reflected: Scheme 3 suffices in achieving the best fit according to the mean deviation, however the variance in achieving this is high. Scheme 2, due to the strong initial oscillations, performs considerably worse than both other schemes. Scheme 3, the truly behavioural one, performs worse than scheme 1 in terms of mean value, but achieves this sightly worse fit at a far lower variance. Since the distance of the mean errors values are more than 6 variances apart from each other, the resulting mean errors improve significantly.

Quantitative results based on the parameter estimates of $R_{3,1}$ are stated in Table 7.2, which in general reflect the qualitative observations. However, the observed biased estimates do not possess a significant influence on the mean value.

Scheme	$\text{mean}(w_3^0 - w_{3,des})$	$\text{var}(w_3^0 - w_{3,des})$
1	-0.0454	0.0024
2	-1.8265	$1.15 \cdot 10^5$
3	-0.0765	0.0004

Table 7.1: Quantitative control performance based on $w_3^0 - w_{3,des}$

Scheme	$\text{mean}\left(\hat{R}_{3,1} - R_{3,1}\right)$	$\text{var}\left(\hat{R}_{3,1} - R_{3,1}\right)$
1	0.069	0.0087
2	0.059	0.2482
3	-0.055	0.0030

Table 7.2: Quantitative estimation performance based on $R_{3,1}$

Based on the quantitative and qualitative results, which indicate that the new scheme in some situations performs better than the compared schemes, the novel scheme for adaptive control in the BF is tested on a mixed time axis.

7.4 Application study - LTV system

7.4.1 Preliminaries

The approach to adaptive control in the BF developed above is designed to operate on a mixed time axis, however to reduce simulation time for the Monte Carlo analysis, a discrete time formulation of the to-be-controlled system was chosen. The setup shown in this section closely resembles a real-world system, where a discrete time estimator/controller setup acts on a continuous-time system.

Plant and controller model class

The system under consideration as the to-be-controlled system is a mass-spring-damper system with an additional adaptive damper, as shown in Figure 7.7. The adaptive damper takes the role of the controller, which is interconnected to the system. The system is assumed to have constant mass m and spring rate c with $m = c = 1$. The damping coefficient b is assumed to deteriorate while the adaptive damper with coefficient b_c acts as the controller.

The plant, without the interconnected damper, i.e. leaving this terminal free, is a continuous time system with $\mathbb{T} = \mathbb{R}^+$, a three dimensional signal space $\mathbb{W}_p = (w_1, w_2, w_3)^T \subseteq \mathbb{R}^3$, which includes the control force as a latent variable. The full plant behaviour is given by

$$\mathcal{P}_{\text{full}} = \left\{ w \in \mathbb{W}_p^{\mathbb{T}} | m\frac{d^2}{dt^2}w_3 + b\frac{d}{dt}w_3 + cw_3 - w_1 = w_2 \right\} \tag{7.19}$$

yielding the dynamical plant model $\Sigma = (\mathbb{T}, \mathbb{W}_p, \mathcal{B}_p)$.

The controller is an adaptive damper, which leads to continuous time formulation on the same time axis as the plant, a two-dimensional signal space $\mathbb{W}_c = (w_2, w_3)^T \subseteq \mathbb{R}^2$ and the controller behaviour

$$\mathcal{C} = \left\{ w \in \mathbb{W}_c^{\mathbb{T}} | b_c\frac{d}{dt}w_3 + w_2 = 0 \right\} \tag{7.20}$$

which leads to the dynamical controller model $\mathcal{C} = (\mathbb{T}, \mathbb{W}_c, \mathcal{B}_c)$.

The corresponding polynomial matrices R_p, M_p and R_c for plant model (in latent variable form) and controller are

$$R_p = \begin{pmatrix} -1 \\ m\frac{d^2}{dt^2} + b\frac{d}{dt} + c \end{pmatrix} \tag{7.21}$$

$$M_p = \begin{pmatrix} 1 \end{pmatrix} \tag{7.22}$$

$$R_c = \begin{pmatrix} 1 \\ b_c\frac{d}{dt} \end{pmatrix} \tag{7.23}$$

The full controlled behaviour of the system (7.19) with an interconnected controller (7.20) is

$$\mathcal{K}_{\text{full}} = \left\{ w \in \mathbb{W}_p^{\mathbb{T}} | m\frac{d^2}{dt^2}w_3 + b\frac{d}{dt}w_3 + cw_3 - w_1 = w_2 \wedge b_c\frac{d}{dt}w_3 + w_2 = 0 \right\} \tag{7.24}$$

which, by elimination, can be brought to the manifest controlled behaviour

$$\mathcal{K} = \left\{ w \in \mathbb{W}_p^{\mathbb{T}} | m\frac{d^2}{dt^2}w_3 + (b + b_c)\frac{d}{dt}w_3 + cw_3 - w_1 = 0 \right\} \tag{7.25}$$

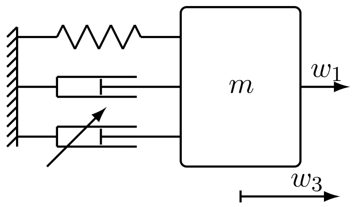


Figure 7.7: Mass-spring-adaptable damper system

The system polynomial matrix of $\mathcal{K}$ is

$$R_{\mathcal{K}} = \begin{pmatrix} -1 \\ m\frac{d^2}{dt^2} + (b + b_c)\frac{d}{dt} + c \end{pmatrix} \tag{7.26}$$

Control of this type does not fit well into the context of regular implementability and related concepts, since the controller is not only interconnected by latent variables, however it is a feasible approach in practice.

Control objective and controller model selection

Similar to the above example, the control objective is to adjust the controller, i.e. the adaptive damper, in such a way that a desired overall damping coefficient b_{des} is achieved. This control objective is achievable if $b \leq b_{des}$.

Since the controller becomes an integral part of the system, the later plant model selection process can only determine the current state of damping of $\mathcal{K}$. To circumvent this restriction of the recursive identification procedure, the adaptive damper is adjusted for the difference between b_{des} and $b + b_c$ with $b_c \geq 0$, which is smoothed by an integrator acting as a low-pass filter. This adaptation mechanism is schematically shown in Figure 7.8.

Plant model selection

The parameter and manifest variable estimation is achieved by making use of an EIV-Kalman filter (EIV-KF) and recursive least squares (RLS) estimator in series, as described above.

The estimated noise-free variables $(\hat{w}_1^0, \hat{w}_3^0)^T$ are used by the RLS estimator to select the model out of the discrete time equivalent of the model class given by (7.19). After fifty seconds, the parameters obtained are used by the

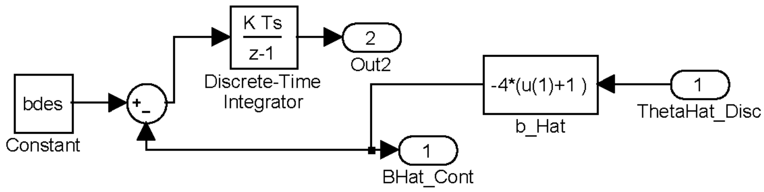


Figure 7.8: Mechanism for adapting the additional damping factor

EIV-KF, while the adaptive control algorithm makes use of the estimated parameters after 100 seconds. The estimator is initialised with a covariance matrix $P_0 = 10^6 I$ and it is configured to adapt to parameter changes with a fixed forgetting factor $\lambda = 0.99$. The calculation of the corresponding continuous time parameters from the estimated discrete time parameters is achieved via Tustin transformation.

7.4.2 Simulation

The system (7.25) is implemented in canonical direct form 2 in order to allow for time varying parameters m, b and c as well as b_c in Simulink. The full plant model is shown in Figure 7.9.

The control system as described above is also implemented in Simulink to allow for a simulation on a mixed time axis, i.e. discrete time for the estimator/adaptation setup and continuous time for the plant and the interconnected damper. The EIV-KF/RLS tandem is implemented as described in Appendix B. The corresponding block diagram is shown in Figure 7.10.

This setup is excited by an external force w_1 in the form of a multiple step signal with randomised magnitudes, satisfying $w_1 \in [-0.5, 0.5]$ and added white uniform distributed noise $w_{1,noise} \in [-0.01, 0.01]$. The simulation is performed over $t_{max} = 500$ s. The damping of the system varies in the form of a sinusoid according to

$$b(t) = 1 - 0.6 \sin\left(\frac{\pi t}{2 t_{max}}\right). \tag{7.27}$$

This shape as well as an estimation result is shown in Figure 7.11, here the green curve shows the true curve while the estimated curve is depicted by

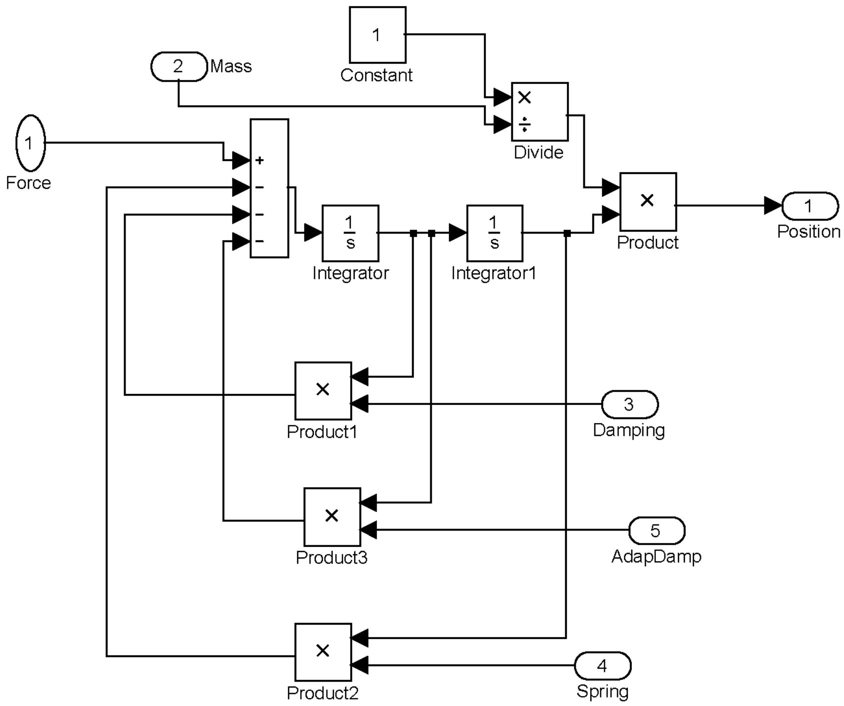


Figure 7.9: Schematics of the system containing plant and controller

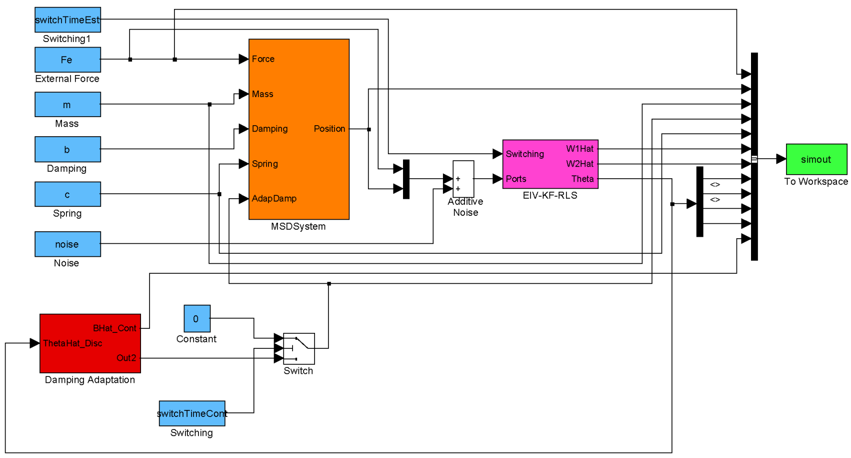


Figure 7.10: Simulink block diagram of the EIV-Estimation/Interconnected Controller setup

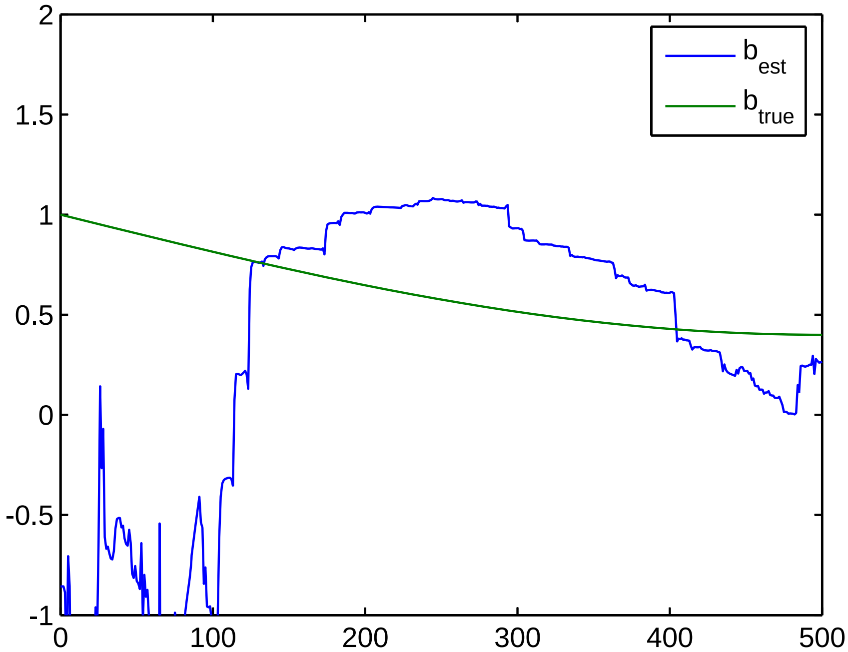


Figure 7.11: True and estimated damping coefficient

the blue graph. It should be noted that this estimate is obtained in closed loop with an additional damper interconnected to the system, therefore the estimate is used to indicate the deviation from the desired value.

Alternatively, a step change in damping after 250 seconds is assumed for the inherent damping of the system, leading from $b = 1$ to $b = 0.5$ according to

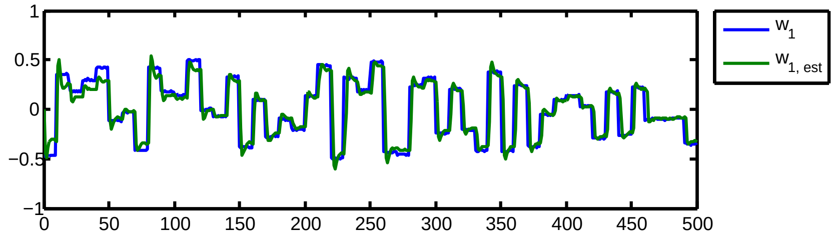
$$b(t) = 1 - 0.5\Theta(t - 250) \tag{7.28}$$

where Θ denotes the Heaviside function.

Both noise free port variables w_1^0 and w_3^0 are corrupted with mutually independent uniform distributed white noise sequences $\tilde{w}_1$ and $\tilde{w}_3$ with a maximum amplitude of 0.01.

7.4.3 Results

The result for a simulation run with $b(t)$ as in (7.27), with a desired damping $b_{des} = 2$, is given in Figure 7.12. Here the blue curve denotes the true position x, the green curve the position estimated by the EIV-KF and the



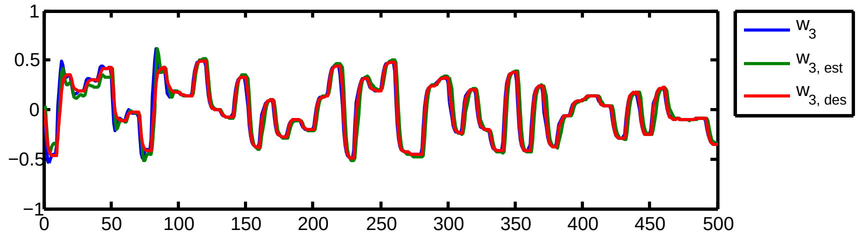


Figure 7.12: True and estimated external forces as well as true, estimated and desired positions for $b_{des} = 2$ and b as specified in (7.27)

red curve indicates the shape of the behaviour that is specified by the adaptive damping. The uncontrolled system is expected to exhibit increasing overshoots, while the selection of b_{des} leads to a critically damped system. This development is stopped by the interconnected controller, instead of overshooting undesirably, the system follows the the red curve closely.

The experiment was repeated for a step change in $b(t)$ as specified in (7.28), her again the lower system inherent $b(t)$ would lead to a large overshoot to be expected in the second part of the simulation. This development is effectively stopped by the controller, here as well the system follows the desired red line closely.

Two further simulation results for runs with $b(t)$ as in (7.27) and with a desired damping $b_{des} = 3$ and $b_{des} = 1.5$ are shown in Figures 7.14 and 7.15, respectively. While the former specifies the desired behaviour to be that of a super-critically damped system, taking long time to reach its steady-state value, the latter exhibits a less than critically damped desired behaviour. The uncontrolled system is again expected to exhibit increasing overshoots. In both cases, the controller is able to maintain the system behaviour closely to the desired one, despite the decreasing $b(t)$.

In all four cases, the proposed control scheme consisting of EIV/KF-

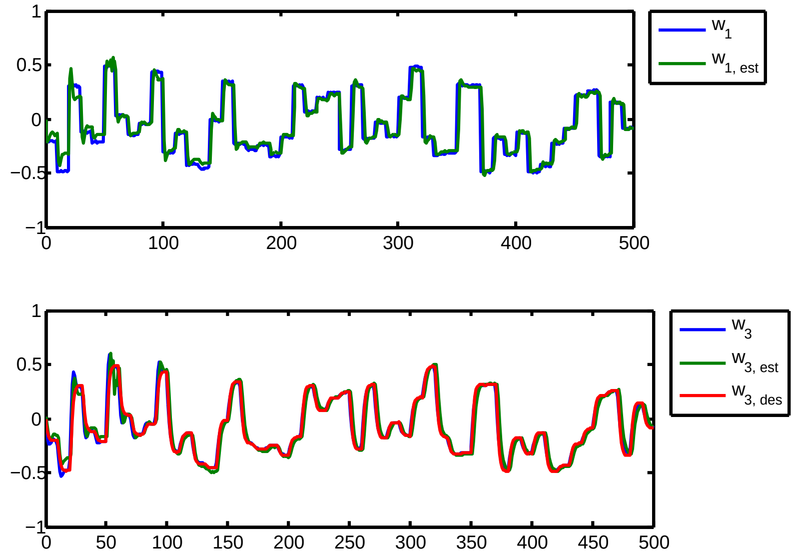


Figure 7.13: True and estimated external forces as well as true, estimated and desired positions for $b_{des} = 2$ and b as specified in (7.28)

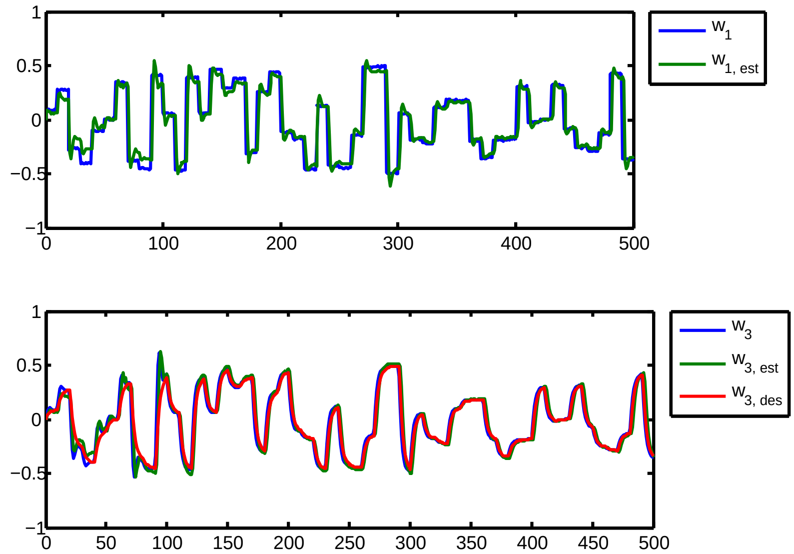


Figure 7.14: True and estimated external forces as well as true, estimated and desired positions for $b_{des} = 3$ and b as specified in (7.27)

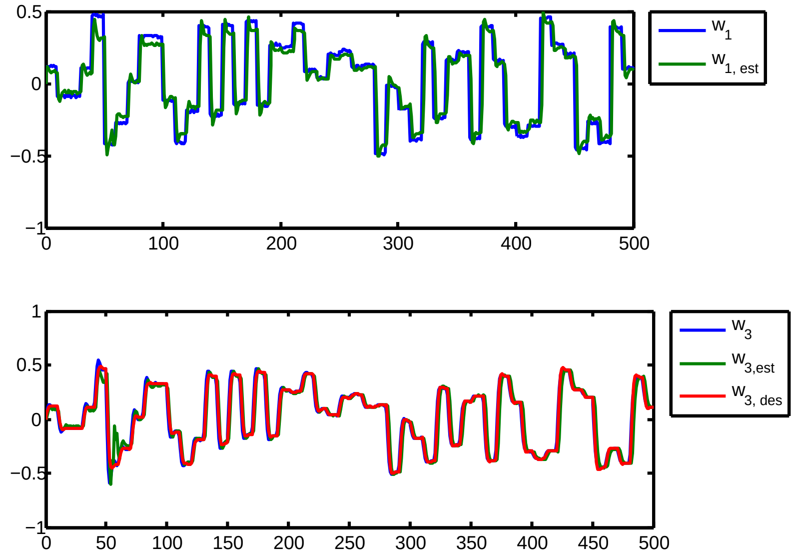


Figure 7.15: True and estimated external forces as well as true, estimated and desired positions for $b_{des} = 1.5$ and b as specified in (7.27)

estimator and interconnected controller was able to maintain the overall system behaviour close to the desired behaviour despite the degradation of the system, be it slowly in a sinusoidal shape or abrupt in form of a step.

7.5 Application study - Bilinear system

7.5.1 Preliminaries

This second application study of the proposed scheme for control in the BF extends the control of an LTV system to that of a bilinear system. The execution on a mixed time axis is maintained, as are the plant and controller model classes and the model selection routine.

Plant and controller model class

The plant is a continuous time bilinear system with $\mathbb{T} = \mathbb{R}^+$ and a three dimensional signal space $\mathbb{W}_p = (w_1, w_2, w_3)^T \subseteq \mathbb{R}^3$, which includes the control force as a latent variable. The full uncontrolled plant behaviour is given by

$$\mathcal{P}_{\text{full}} = \left\{ w \in \mathbb{W}_p^{\mathbb{T}} | m \frac{d^2}{dt^2} w_3 + b \frac{d}{dt} w_3 + c w_3 + d w_1 \frac{d}{dt} w_3 - w_1 = w_2 \right\} \quad (7.29)$$

where d denotes the bilinear coefficient, which is assumed to express a multiplicative effect from external force and position, e.g. a nonlinear spring. This leads to the dynamical plant model $\Sigma = (\mathbb{T}, \mathbb{W}_p, \mathcal{B}_p)$. The model was tuned such that it exhibits a behaviour close to the limit of stability in the simulation, which is an undesired behaviour to be avoided by the interconnected controller.

As in the above example, the controller is an adaptive damper, which leads to continuous time formulation on the same time axis as the plant, a two-dimensional signal space $\mathbb{W}_c = (w_2, w_3)^T \subseteq \mathbb{R}^2$ and the controller behaviour as in (7.20), yielding $\mathcal{C} = (\mathbb{T}, \mathbb{W}_c, \mathcal{B}_c)$.

The full controlled behaviour of the system (7.19) with an interconnected controller (7.20) is

$$\mathcal{K}_{\text{full}} = \left\{ w \in \mathbb{W}_p^{\mathbb{T}} | m \frac{d^2}{dt^2} w_3 + b \frac{d}{dt} w_3 + c w_3 - w_1 = w_2 \wedge b_c \frac{d}{dt} w_3 + w_2 = 0 \right\} \quad (7.30)$$

which, by elimination, can be brought to the manifest controlled behaviour

$$\mathcal{K} = \left\{ w \in \mathbb{W}_p^{\mathbb{T}} | m \frac{d^2}{dt^2} w_3 + (b + b_c) \frac{d}{dt} w_3 + c w_3 + d w_1 \frac{d}{dt} w_3 - w_1 = 0 \right\} \quad (7.31)$$

Since the controller cannot adequately address the bilinear behaviour of the plant, it has to compensate for this effect by adaptation of its damping factor. For this purpose, the plant model class is a recursively updated kernel representation model of order 2.

Control objective and controller selection

The control objective is to achieve a behaviour as close as possible to a critically damped linear mass-spring-damper system. The controller is selected as above for the LTV system. For this purpose, the current damping coefficient (under the assumption of an LTV system) is estimated and the adaptive damper set to the respective value.

Plant model selection

The plant model is selected by making use of the same scheme as above, all parameters remain the same. The plant model is estimated in the form of a second order kernel representation, in this way, the misfit between the bilinear plant and the linear model is compensated for in the model selection process. In this way, the plant model is appropriate for the selection of the controller that achieves the closest possible behaviour to the desired one.

7.5.2 Simulation

Beside the implementation of the bilinear plant, all implementation details remain unchanged in comparison to the LTV study conducted above. In order to be able to show the uncontrolled behaviour, the experiment is executed over a timespan of 1000 s, of which the first 500 s the plant is left without control, while after 500 s the controller takes action.

The parameter set is selected to be $m = c = 1$, $b = 2$ and $d = 4$. The parameter set is kept constant over each experiment, as the aim is to show the capability of the proposed scheme to handle nonlinear systems in addition to LTV systems. The desired critically damped behaviour requires a damping coefficient $b_{des} = 2$ for the linear equivalent. In addition to adding this damping constantly to the system, it is necessary to adapt the damping to the nonlinearity.

7.5.3 Results

The plot of the resulting behaviour is shown in Figure 7.16, form this graph the state dependent stability and behaviour of the system becomes obvious for the uncontrolled part upto $t = 500$. After the controller takes action ($t > 500$), similar amplitudes of excitation do not lead to oscillations, which indicate the stability margin. In the upper range of the operating range, no stability problems are obvious and the damping is already appropriate due to the chosen $b = b_{des}$. In this part of the operating range, the controller, which is limited to a positive damping coefficient, cannot improve the behaviour.

The overall performance of the adaptive control system is well, taking into account the highly nonlinear behaviour of the plant, the behaviour of the controlled system is far closer to a linear one and is damped almost critical, still slightly depending on the operating point.

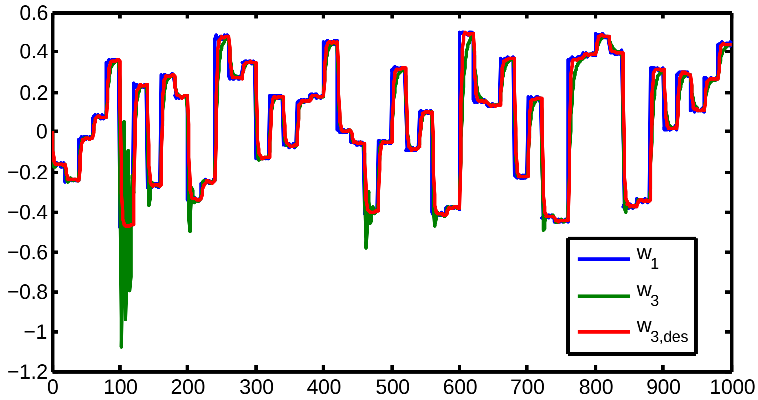


Figure 7.16: Actual and desired positions (w_3, $w_{3,des}$) as well as exciting force (w_1) for a bilinear plant

7.6 Concluding remarks

Control in the BF is achieved by interconnected control systems, that transfer their control laws via commonly used variables of plant and controller. This type of control is inherently robust due to its insensitivity to noise. The treatment of control in the BF in literature is mostly limited to LTI systems, while in practical applications nonlinearities or time-varying systems are frequently encountered.

The current state of research on adaptive control follows the proposal of Willems (Willems, 1986b) to apply the MPUM for adaptive control purposes, as described in (Polderman, 2000; Polderman and Mareels, 1999). The limitations of practical applications are not taken into account in current literature, which includes the assumption of the availability of noise-free system variables.

The approach chosen in this work is different from previous ones in that it breaks down the problem of adaptive control into smaller subproblems, a scheme for adaptive control in the BF is developed, clearly aiming on an applicable formulation. The potential application to practical systems is addressed by considering conditions prevalent in practical applications of control, such as finite processing time of computer control systems, limited controller availability or the unavailability of latent variables for measurements.

At the core of the novel scheme is an EIV-KF/RLS tandem estimator for model selection, together with some data postprocessing, selects an adequate model under the symmetric noise conditions. This scheme is different from existing approaches by using a grey-box approach to limit the plant and controller model class, instead of using the MPUM. The focus on applicability to real world problems is highlighted by the provision of implementation ready SIMULINK components of the controller, which can be easily ported to real world controllers

The novel scheme is tested in comparison to two alternative approaches marking different levels of application or acceptance of the BF, this comparison is carried out with a plant simulated in discrete time by making use of a Monte Carlo analysis. It turns out that, under the conditions of the study, the proposed scheme performs adequately and mostly better than the compared schemes.

In very close resemblance of a practical setup, the proposed scheme is is applied to a mixed time axis problem of a degrading damper, which is solved adequately for different desired behaviours. It also performs well in linearising a bilinear system by making use of the same adaptive damper. This application of the novel scheme may find some interest in control applications where higher level control systems can better handle linear systems. Put into such applications, the adaptive control by interconnection would become part of the nonlinear to-be-controlled system, which yields a linear subsystem to higher order system, but is robust and energy efficient due to the direct integration into the system.

The work presented in this chapter can be extended and made more significant by application to more complex plant models formulated in an acausal simulation environment, in this way testing the performance of the setup under inversion of the causal direction. As a further step, applications to real world plants have the potential to indicate the applicability and acceptance for control tasks.

Chapter 8

Conclusions and Further Work

> Although to penetrate into the intimate mysteries of nature and thence to learn the true causes of phenomena is not allowed to us, nevertheless it can happen that a certain fictive hypothesis may suffice for explaining many phenomena.
>
> Leonhard Euler

8.1 Conclusions

The motivation for undertaking the research documented in this thesis stems from the hypothesis that the Behavioural Framework, when applied to practical control engineering problems, yields potential for further optimisation of modelling and control performance. This further performance increase is necessary in times of diminishing primary energy resources and increasing world population, in order to increase energy efficiency.

The analysis of the potential for application of the Behavioural Framework has been conducted following typical control engineering tasks, supported by simulations of systems in simulation software. In this way, comparable results have been provided together with example applications, that may serve as a basis for further applications.

As a basic task of control engineering, the derivation of models from physical principles has been treated, in conjunction with the proposal for two different graphical representations, each suitable for a specific range of applications. This has been carried out in the perspective of practical considerations related to models, especially validation and verification.

From the analytical modelling task, the validity and interpretability of models has been considered. Potential pitfalls have been highlighted. The validity of models has been extended by representation and application of bilinear models in the Behavioural Framework, following an analysis of selected nonlinear model classes and of bilinear systems in particular.

The existing approaches to approximate modelling have been reviewed and a proposal for extension has been followed, leading to the development of a practical approach to approximate modelling. This approach has been tested in comparison to other approaches as well as in application to real world systems and data.

Proposed approaches to adaptive control in the Behavioural Framework have been analysed and a practically viable approach has been formulated. This approach takes into account limitations encountered in practical control applications. It has been tested in comparison to other approaches and on mixed time axis models of linear time varying and bilinear systems.

8.2 Main outcomes of the research

As a main outcome of the research, the testing of the applicability of the Behavioural Framework on selected practical engineering tasks is considered. The results indicate an advantage in most cases, which may lead to further applications. This main outcome has been achieved by partial extension of the Behavioural Framework in areas relevant for control engineers.

The most relevant single extension of the available behavioural techniques is the development of a scheme for adaptive control. This scheme has been developed with the limitations encountered in practical control problems in mind, which leads to it being ready for implementation on computer control systems. The implementation readiness of the approach is highlighted by its availability as SIMULINK blocks which may be easily ported. The availability of such a scheme may eventually lead to more adaptive control being implemented in the form of interconnected controllers.

Another important result has been the development of an approach to approximate modelling, following a proposal by (Willems, 1987), that resembles closely the action of a control engineer in weighing misfit and sensitivity or lag of the resulting model. This algorithm, while it is not as generic as the original algorithms for approximate modelling, is formulated in algorithmic form and tested on real world data and realistic system models.

Last but not least, depending on the field of application of the reader, the representation and analysis of bilinear systems in the Behavioural Framework forms an important step in a framework dominated by the assumption of linearity and time invariance. The analysis has been performed both analytically and by simulations, which makes it accessible for both applied mathematicians and control engineers.

8.3 Summary of contributions

The research related to this thesis has led to contributions to the body of knowledge, which are outlined below in descending order of significance considered by the author.

- Development of a scheme for adaptive control in the Behavioural Framework: a scheme for adaptive control has been developed and tested in comparison to other approaches as well as on practical mixed time axis systems. Part of this contribution is the high degree of implementation readiness. This contribution is the subject of Chapter 7.

- Development of a combined misfit/latency approach to approximate modelling: a combined approach has been developed and tested on practical modelling problems. This approach, which has been proposed by (Willems, 1987) in similar form, is formulated as an algorithm. The development is described in Chapter 6.

- Representation and analysis of bilinear systems in the Behavioural Framework: a suitable representation for bilinear systems in the Behavioural Framework, the so called bilinear extended kernel representation, has been derived. The existence and uniqueness of the solutions to the respective behavioural equations has been shown and the practical applicability of the novel representation has been tested. This is the subject of Chapter 5.

- Establishment of a link to graphical representations: the links to two appropriate graphical model representations for application in the Behavioural Framework have been established, their respective areas of application have been identified and examples have been presented in Chapter 4.

- Analysis of nonlinear model classes: the applicability of commonly used nonlinear model classes to the Behavioural Framework has been analysed and an appropriate class identified. This is documented in Section 5.2.4.

- Analysis of software tools: software tools capable of simulation in an acausal environment have been identified and reviewed following practical requirements. This helps in the selection of software when approaching a practical control engineering problem, the results are documented in Section 3.4.

- Identification of material for curricula development: the material, by which the standard control engineering curriculum has to be extended to ease access to the Behavioural Framework, has been identified and described in Sections 2.2 and 2.4.

- Application of behavioural techniques to practical problems: the applications of behavioural techniques to practical control problems presented throughout this thesis serve as examples for further applications.

8.4 Proposals for further work

Bearing in mind the aim of this work, the main proposal for further work is the application of behavioural methods, including those developed in this work, to real world engineering problems. The practically relevant models of systems used in this work are suitable to give an indication, but the ultimate test of a framework has to be the application in industrial practice.

A further step in the refinement of the scheme for adaptive control would be the test on a plant formulated in an acausal modelling and simulation language, e.g. Modelica, to test the reaction of the overall control system to a system changing the causal directions. While the applied controller and model structures are intended to be used in acausal settings, a series of tests will indicate the performance of the algorithm.

This extension can be performed relatively easy by exchanging the plant in the associated SIMULINK diagram by a SimScape formulation of an appropriate plant, by making use of the already available SIMULINK blocks implementing the control setup. It is then further required to adapt the current excitation structure such that an excitation by force or displacement can be achieved.

The approximate modelling in terms of the extended bilinear kernel structure was achieved using a TLS estimator. There exist better suited algorithms for EIV parameter estimation of bilinear models, e.g. those proposed in (Larkowski, 2009). A common application of these EIV techniques and the behavioural techniques developed in this work may require some adaptation of the identification algorithms, but will likely prove effective.

The use of such novel EIV identification techniques in the combined misfit/latency approach to approximate modelling is also very likely improve the performance of this approach. To achieve this, the misfit measure currently obtained from the singular values of the observation matrix, will have to be adapted for the novel EIV techniques.

A feasible approach would be to make use of the Koopmans-Levin method, which can be extended to a generalised singular value decomposition (Vajk, 2005). This decomposition can then be arranged such that the singular values contain misfit data, which can be used in the algorithm to select the lag structure of the model.

The application of the bilinear extended kernel representation to control purposes in the Behavioural Framework can be expected to yield efficiency improvements, similar to those discussed in (Martineau *et al.*, 2004). To achieve this control setup, a bilinear control structure suitable for interconnection to the manifest variables needs to be developed. Here the adaptation mechanisms developed for the adaptive control system may be considered useful to update the bilinear control structure.

The selection of course material for a possible curriculum extension has not been brought into a didactically suitable form, which will enable its introduction in the form of a short course. This presentation of the material, together with appropriate exam or coursework questions and a feedback form will provide further evidence of the applicability of the Behavioural Framework.

Of a more theoretic interest is the question for suitable and practically feasible complexity measures, perhaps even to overcome the difficulties in comparing different model classes, as noted in Remark 1. This extension could be based on the development of a complexity measure other than by mutual inclusion, possibly based on the comparison of polynomial approximations of the differential equation. In this way, different classes of equations can be mapped to one vector space, which than can be ordered accordingly.

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Appendix A

Modelica Implementation of a Bilinear System

Introduction

The acausal simulation software packages employed in this thesis, SimScape (The Mathworks, Inc., 2012b) and MapleSim (Maplesoft, 2012), offer in their standard libraries mostly linear, sometimes parameter varying model elements.

For this reason, it is necessary to either implement the bilinear term either by help of an additional source of effort, e.g. a force element, or implement a specialised block in the Modelica language. The former variant would not result in a fully acausal bilinear system, since the specification of the effort to be applied has to be performed in causal signal transformation blocks. For this reason, the implementation of a particular Modelica block is the option chosen.

Equation development

The system is assumed to be a mechanical translational second order system formulated on a continuous time scale $\mathbb{T} = \mathbb{R}$. The system has two mechanical translational ports, with the force F being the common variable to both ports and both ports having a different position, x_0 and x_1, respectively.

The system can be considered as a spring-damper system with a non-negligible mass and a bilinear interconnection between between velocity and time derivative of the force.

The signal space of manifest variables is $\mathbb{W} = \{x_0, x_1 F\}$ and the behaviour is governed by the bilinear differential equation

$$\begin{aligned} F(t) = c\,(x_0(t) - x_1(t)) + b\frac{d}{dt}\,(x_0(t) - x_1(t)) + m\frac{d^2}{dt^2}\,(x_0(t) - x_1(t)) \\ + K\frac{d}{dt}\,((x_0(t) - x_1(t))\,F(t)) \end{aligned} \tag{A.1}$$

for a mass m, damping coefficient b, spring rate c and the bilinear coefficient K.

Modelica code

The Modelica code is generated by help of the MapleSim tool for custom generated Modelica components, in this way it is only necessary to specify parameters and the equations of the system.

```
model BilinearTranslational
    parameter Real c = 1 "c";
    parameter Real b = 1 "b";
    parameter Real m = 1 "m";
    parameter Real K = 1 "K";
    Real F0;
    Real x1;
    Real diff_msim_x0_1;
    Real diff_msim_x1_1;
    Real x0(start = 0);
    Modelica.Mechanics.Translational.Interfaces.Flange_a tflange
    Modelica.Mechanics.Translational.Interfaces.Flange_a tflange0
equation
     F0 = c * (p0 - p1) + b * (diff_msim_p0_1 - diff_msim_p1_1)
     + m * (der(diff_msim_p0_1) - der(diff_msim_p1_1))
     + K1 * ((diff_msim_p0_1 - diff_msim_p1_1) * diff_msim_F0_1
     + (p0 - p1) * der(diff_msim_F0_1));
    diff_msim_F0_1 = der(F0);
    diff_msim_p0_1 = der(p0);
    diff_msim_p1_1 = der(p1);
```

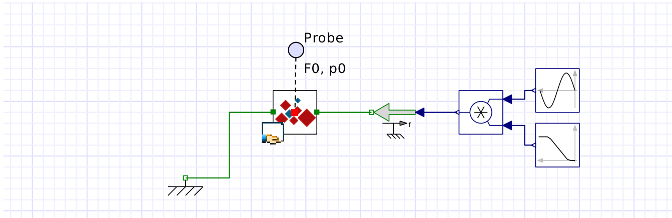

Figure A.1: Simulation setup for bilinear system in MapleSim

```
    tflange.f = F0;
    tflange.s = p1;
    tflange0.f = F0;
    tflange0.s = p0;
end BilinearTranslational;
```

The Modelica code is than executed by help of a block, linked to the Maple file containing this equation and parameter definition, any updates to the equations can be transferred directly to MapleSim. The simulation setup is shown in Figure A.1.

Appendix B

EIV-KF S-function implementation

Preliminaries

In order to be able to use the KF/RLS tandem estimator in a simulation on a mixed time axis, an implementation suitable for execution in an adequate tools is required. For an RLS estimator, this is a standard step, however for the EIV-KF, this implementation is described in the sequel.

Due to the ease of porting such implementations to real world control systems, as a basis the Simulink block diagram software and the related m-file S-function is chosen. While this software in principle does not allow for acausal simulation, it can be extended by use of the appropriate add-on packages from the SimScape product range. On the other hand, especially the implementation of mixed time axis examples is better supported by Simulink.

EIV-KF S-function

The implementation consists of a number of setup steps, in which mainly the time behaviour, the workspace variables and the parameters are defined, the steps are omitted for brevity.

The block has three input variables, namely the estimated parameters $\hat{R}$, the noise variance σ and the measured port variables w_i, $i = 1, 2$. It has one output variable, the estimated noise free port variables, $\hat{w}_i^0$, $i = 1, 2$.

During operation of the EIV-KF block, first the initialisation routine is called and assigns the initial values to the work space variables. During each recursive update of the block in the Simulink model is executed, calling the function `Update(block)`, which effectively runs the EIV-KF in three steps. The stored data is saved in the work space variables `block.Dwork(`n`).Data`, since no common data storage beside these is available to have values for the next recursive step. The output values are created from the workspace variables after the recursive update in the function `Output(block)` and provided to the next system, in this case the RLS.

```
function eivKF(block)
setup(block);

function Start(block)
 % Initialisation of work variables
  block.Dwork(1).Data = rand(1,4);
  block.Dwork(2).Data = [0 0 0 0 0 0];
  block.Dwork(3).Data = 0;
  block.Dwork(4).Data = [0 0 0 0 0 0];

function Outputs(block)
  block.OutputPort(1).Data = block.Dwork(4).Data(end-1:end);

function Update(block)
  % EIV-KF
  %Initialisation
  s = block.Dwork(2).Data';
  k = block.Dwork(3).Data;
  R = [block.Dwork(1).Data(1:2)';block.Dwork(1).Data(3:4)'];
  Sigma = diag(repmat([block.InputPort(2).Data.^2]',1,3));

  AA = s'*s;
  BB = [R(end-1:end, end-1:end), zeros(2, 2); zeros(2, 4)];
  R = BB(end-1:end, end-1:end)+ k*Sigma(end-1:end, end-1:end)*...
      AA(end-1:end, end-1:end);
  block.Dwork(1).Data = [R(1,:), R(2,:)];
```

```
a = block.InputPort(1).Data(1:2:3);
b = block.InputPort(1).Data(2:2:4);
mbar = [1, 0, a(1), b(1), a(2), b(2)];
R = [block.Dwork(1).Data(1:2)';block.Dwork(1).Data(3:4)'];
Sigma = diag(repmat([block.InputPort(2).Data.^2]',1,3));
%Step 2
s = mbar*[(eye(2) - R), zeros(2, 4); zeros(4, 2), eye(4)];
k = 1/(s*Sigma*mbar');
block.Dwork(2).Data = s;
block.Dwork(3).Data = k;
%Step 3
eta = -mbar*[block.Dwork(4).Data(end-3:end); ...
    block.InputPort(3).Data];
block.Dwork(4).Data = [block.Dwork(4).Data(end-3:end); ...
    block.InputPort(3).Data]+k*Sigma*s'*eta;
```

Appendix C

Brief History of Control Engineering

C.1 Early stages

The early stages of control, as defined in (Bissell, 2009) by the use of feedback, employed flow control for water clocks. This knowledge originated mainly in the Arab and Hellenic worlds. Despite translations of the works of Ktsebios and Heron being available in Europe, the work was reinvented in the 18th century by Cornelius Drebbel. Drebbel conceived the first temperature regulator, consisting of a fluid thermometer controlling the flue of a furnace and already featuring screws to adjust the setpoint temperature.

Following flow and temperature regulation, the next step in sophistication was the speed regulation of windmills. The problem was solved by the mill fantail, adjusting the mill sails into the wind direction and regulating the speed of the mill via the angle of attack of the mill sails. A second control challenge in the control of mills was the distance between the millstones, which increases with rotational speed of the stones, resulting in lower quality flour being produced. The most ingenious systems to increase the force pressing the millstones together in this case was Thomas Mead's *lift-tenter* using a centrifugal pendulum, resembling the centrifugal governors applied to steam engines.

Steam engines emerged around 1780 as reciprocating engines for pumping water which were not in need of velocity regulation, while rotational steam engines as prime movers were sold about ten years later. Matthew

Boulton, collaborator of James Watt, was inspired by a lift-tenter which lead to the invention of the centrifugal governor as a means of speed regulation of steam engines around 1790. This device followed the rapid spread of the steam engine throughout Europe in the early 19th century.

The increasing numbers of centrifugal governors, designed not only by the inventors Boulton and Watt, made the problems of this type of regulator obvious:

- Lacking integral action, a centrifugal governor could not compensate a steady-state error.
- The time required to compensate a disturbance was long.
- Nonlinear frictional forces lead to limit cycling, in the 19th century termed *hunting*.

To overcome the problem of offset in the control, elaborated mechanical systems such as the Siemens chronometric governor were conceived, mostly suffering from stability problems.

For the tuning of the different governors with respect to steady-state behaviour, graphical methods were developed in the 19th century, however the number of engineers and scientist working on the dynamic behaviour was small. The first demonstration of possible instability of the regulator was put forward by George Bidell Airy, leading the way to Maxwell's classic paper *On Governors* (Maxwell, 1867), where Maxwell derives a third order linear model and conditions for stability of this model and asks support from mathematicians for higher order models. The support asked by Maxwell lead to Routh developing his version of the Routh-Hurwitz stability criterion, published in 1877 (Routh, 1877).

On Governors employs formulations compatible with those of the behavioural framework, especially control targets are formulated in terms of qualitative behaviours and no notion of input or output is employed throughout. Also an interconnected control of differentiating type is recommended to stop oscillations from limit cycling.

C.2 Control engineering in the early 20th century

Towards the end of the 19th century and in the first decades of the 20th century, control is employed outside velocity and flow regulation problems.

The first different applications were autopilots for maritime use and gun turret hydraulics, soon leading to the gyroscopic action and servomotors used in ships and later torpedoes.

The same term also saw cascaded control loops in the form of Sperry's autopilot as well as adaptive control under the term *anticipator* and the first text books on regulation of prime movers (Tolle, 1905). The first three-dimensional aircraft autostabilizers by Elmer and Lawrence Sperry was characterized as (Bennett, 1993, p. 137):

> 'The system was normally adjusted to give an approximately deadbeat response to a step disturbance. The incorporation of derivative action - the equivalent of 'meeting' the helm - was based on Sperry's intuitive understanding of the behaviour of the system, not on any theoretical foundations.'

The above quote shows that while theoretical foundations were far developed, control engineering was dominated by practising control engineers.

Another parallel strand of development was originating from the growing field of telecommunications, creating an increased interest in operational amplifiers and electrical circuits. The work of communications engineers such as Harold Black, Harry Nyquist and Hendrik Bode was built on sound theoretical foundations, suitable for the systems under consideration. Communications engineering aims at signal processing, thus an input and output can be assigned to the overall system from an application point of view. Further to this, several components of the circuit such as operational amplifiers or transistors are designed to behave in defined input-output settings in their operating range.

The sound theoretical basis layed by communications engineers in the first decades of the 20th century, formulated with their particular problems in mind, was soon distributed and accepted worldwide.

C.3 Second World War

Before and during World War II (WWII), in the four dominating forces United States, United Kingdom, Soviet Union and Germany, control was considered to be a key factor in the development of weapons for this new kind of war. Especially in the US, considerable effort was generated by granting support and gathering researchers in appropriate groups under military control. In the UK, pre-war efforts were shared between government bodies and companies, but not centrally organised. The same applies to Germany, were strong distributed efforts were taken, but the general importance of the area of control engineering was only observed by the professional body *Verband Deutscher Ingenieure VDI*, founding a specialist committee as early as 1939. One particular interesting publication by Herman Schmidt (Schmidt, 1941) establishes the link between control engineering, economics and social sciences. Schmidt later received a call for the first chair of control engineering.

In the Soviet five-year plans of the time, industrial applications of control were considered important and the universities of Moscow and Gorkii became centres for control engineering in the Soviet Union.

These bases lead to an increasing amount of theoretical methods, initially applied secretly. Most notably among those techniques were the notion of sampled data systems and the z-Transform. The theoretical research however was, especially in the US, split into the three groups process control, feedback amplifiers and servomechanisms. Each of these groups developed and used its own language and frameworks, despite the efforts taken to standardise language and enable interdisciplinary exchange. The confidentiality of the results during war time lead to a vast number of theoretical papers being published after the war.

C.4 After World War II – The Gap

During the war, theoretical studies were conducted into ever more abstract topics, while the engineering education was still lagging behind, resulting in what Bissell (Bissell, 1994, p. 149) describes in the following statement:

> 'After the war, the comparatively small circle of engineers who had become adept at what we now call 'classical control' and 'the systems approach' found that they could be hardly understood by many of their colleagues.'

Before the war, the mathematics education of engineering students comprised arithmetic, geometry, trigonometry, algebra and analytical geometry while calculus, differential equations and further advanced topics were not considered useful for engineers. This changed after WWII, bringing subjects such as ordinary differential equations, Fourier series and stochastics into the canon of major universities. At the same time, research in automatic control is moved from the laboratory and rigour of the theories is considered more valuable than application on a physical system. This movement was seen as a *gap* by George Axelby (Axelby, 1964) and efforts were taken to close it.

In parallel to this shift towards theory, practising control engineers found the view of a system as a signal manipulator, originating from the work of the feedback amplifier strand of control engineering history, well applicable and especially less specific to the type of system. This period can be considered the time of introduction of *a priori* causality into the systems domain (Willems, 2007b). Together with this change of paradigm, the graphical modelling techniques changed towards block diagrams and equivalent circuits, using exclusively the notion of an input-output relation instead of a system. This abstraction of modelling and the wide availability of simulation technology is considered to widen the gap between practice and theory (Bergbreiter, 2005, p. 35), however it also plays an important role in the economic application of control techniques.

Out of a conference called *Regelungstechnik - Moderne Theorien und ihre Anwendbarkeit* (Control engineering - modern theories and their applicability) organised in 1956 by the German VDE/VDI control subgroup, an initiative to found an international organisation for control lead to the constitution of the International Federation of Automatic Control (IFAC) in 1957.

Around 1960, scientists from the Soviet Union went from the transfer function to the state space approach, a further shift in paradigm bringing control closer to mathematics. Still after this extension of models to input/state/output models, not all modelling problems could be tackled in this framework, leading to the development of a further framework to view systems and control, the behavioural framework put forward by Jan C. Willems (Willems, 1979). Despite its general recognition as being the right framework for certain modelling and control tasks, it is mainly worked on from a theoretical engineering or applied mathematics point of view. This may be due to the same circumstances that led to the quote at beginning of this section, since the mathematics underlying the behavioural framework is again shifting. The behavioural framework strongly uses set and group theory, theoretical aspects of differential equations and measure theory, which is rarely taught to control engineers despite their good mathematics foundations. A recent movement towards acausal modelling and simulation, originating from practical applications, can be observed in applied publications and interest in acausal simulation tools such as implementations of the Modelica language (Modelica Association, 2012) or bond graph based modelling such as 20-sim (Controllab Products B.V., 2012).